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# Portfolio Optimization and Ambiguity Aversion

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**Abstract**

This thesis analyses whether considering ambiguity aversion in portfolio optimization improves the out-of-sample performance of portfolio optimization approaches. Furthermore, it is assessed which role ambiguity aversion plays in improving the portfolio performance, especially compared with the role of estimation errors. This is done by evaluating the out-of-sample performance of the approach of Garlappi, Uppal and Wang for an investor with multiples priors and aversion to ambiguity compared to other portfolio optimization strategies from the literature not taking ambiguity aversion into account. It is shown that considering ambiguity aversion in portfolio optimization can improve the out-of-sample performance compared to the sample based mean-variance model and the Bayes-Stein model. However, the minimum-variance model and the model of naïve diversification, which are both independent of expected returns, outperform the approach considering ambiguity aversion for most of the empirical applications shown in this thesis. These results indicate that ambiguity aversion does play a role in portfolio optimization, however, estimation errors regarding expected returns overshadow the benefits of optimal asset allocation.

*Keywords:* portfolio choice; asset allocation; estimation error; ambiguity; uncertainty.

## 1. Introduction

### 1.1. Problem definition

The classical mean-variance portfolio optimization approach, established by Harry Markowitz, is based on the assumption that expected asset moments are known. In real-life investment decisions, these input parameters are, however, unknown and need to be estimated. Typically, the input parameters are estimated from realized returns and as a result, they are inevitably estimated with error. This is particularly important because the choice of input parameters has a huge effect on the optimal portfolio weights and consequently on the portfolio performance. As these estimation errors are ignored in the classical portfolio optimization approach, a bad out-of-sample performance is the result. Mean-variance optimized portfolios often show a low degree of diversification and extreme portfolio weights[1]. To improve the performance of the classical mean-variance approach, new methods were developed that explicitly take into account that the estimated asset moments are not the true values. In this context, a very important class of methods is the class of Bayesian approaches. Using these approaches, the input parameters are adjusted before they are used to determine the optimal portfolio. This is done by combining a priori information with the historical data by the means of Bayesian updating. Empirical studies have shown that these adjustments lead to a better out-of-sample performance (e.g. Jorion, 1986).

Bayesian approaches, however, assume that the probability distribution of outcomes is known and the decision-maker has a unique prior for the outcomes, while it is ignored that this is not the only possible probability distribution. From the uncertainty about the true probability distribution, a new type of uncertainty arises, called ambiguity. In this thesis, ambiguity is defined as Knightian uncertainty (Knight, 1921), which refers to uncertainty that can't be measured because the probability distribution is unknown and risk refers to measurable uncertainty with a known probability distribution. In 1961, Ellsberg was the first to show experimentally that investors are averse against this type of uncertainty (Ellsberg, 1961). He showed that the majority of investors prefer a risky investment to an ambiguous investment. A central Bayesian assumption is that all uncertainties can be reduced to risks. This assumption, however, contradicts Ellsberg's empirical results. If an investor is ambiguity

[1] See for example Hodges and Brealey (1973), Best and Grauer (1991) and Black and Litterman (1992)

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averse, his behavior in the Ellsberg experiment leads to a violation of Savage's subjective expected utility theory (Savage, 1954), because the sure-thing-principle is disobeyed. Therefore, ambiguity averse behavior is not rational according to subjective expected utility theory. As a consequence, new portfolio optimization models were developed that allow for ambiguity aversion by weakening the sure-thing principle[2].

In the center of this thesis is the multi-prior approach introduced by Garlappi et al. (2007). They developed an approach that takes both, parameter uncertainty about the true asset moments and ambiguity aversion into consideration. They did so by extending the classical mean-variance approach by two main points. Firstly, they implement an additional constraint on the expected return for each asset to lie within a confidence interval of the estimated expected return. Secondly, they apply an additional minimization over the choice of possible expected returns. Ambiguity aversion is implemented by this additional minimization. Furthermore, the model of Garlappi, Uppal and Wang is also able to account for model uncertainty. This is in case the investor forms his beliefs about expected returns from a factor model, like the Capital Asset Pricing Model (CAPM), but is unsure whether this factor model is the true return-generating model. Garlappi, Uppal and Wang present in their empirical study, that accounting for ambiguity aversion, when optimizing portfolios, improves the portfolio performance and leads to a higher out-of-sample Sharpe ratio compared to the classical mean-variance approach and the Bayes-Stein approach.

This thesis focuses on the question whether the out-of-sample performance can be improved if ambiguity aversion is considered in portfolio optimization, and which role ambiguity plays in improving the portfolio performance. To answer these questions, an empirical study is performed to compare the performance of the ambiguity-averse approach introduced by Garlappi, Uppal and Wang with the performance of several other models from the literature, like the classical mean-variance model, the minimum-variance model, the Bayes-Stein model and the model of naïve diversification. Additionally, the results are tested for robustness by varying important model assumptions. In this way, it is analyzed which additional value can be generated by taking ambiguity aversion into consideration. It is of particular interest which role ambiguity aversion plays compared with the role of estimation errors. Furthermore, the relationship between risk aversion and ambiguity aversion is analyzed and its role for the optimal portfolio is discussed. Besides investigating the effect of ambiguity aversion on the out-of-sample performance of optimized portfolios, a critical view is taken on the topic. Crucial issues, like the rationality of ambiguity averse investment decisions and learning under ambiguity, are discussed.

[2] See for example Schmeidler (1989) and Gilboa and Schmeidler (1989)

### 1.2. Method of investigation

To analyze the effect of ambiguity averse investment decisions on the portfolio performance, an empirical study is performed. This empirical study is based on the work of Garlappi, Uppal and Wang and is divided into three parts. In the first part, the original results of Garlappi, Uppal and Wang are replicated with a special focus on differences in results. With regard to this empirical application, the investor can build his portfolio from eight international equity indices. In the second part, important input parameters like the sample size to determine the optimal portfolio weights, the degree of risk aversion and the timespan are varied. Finally, in the third part of the empirical study, the investment universe is changed and the portfolio optimization problem is applied to the German DAX30 stocks. Most assumptions are thereby taken from Garlappi, Uppal and Wang to ensure the comparability of the results. This empirical design is chosen to show whether incorporating ambiguity aversion to the portfolio optimization problem leads to a better out-of-sample performance independent of the framework conditions. Furthermore, the variation of input parameters ensures that the improved out-of-sample performance results from taking ambiguity aversion into account and not from other characteristics, for example time-specific or asset-specific characteristics.

The rest of this thesis is organized as follows. In Chapter 2, the foundations of modern portfolio theory are presented and the problems arising from the classical mean-variance approach are discussed. Subsequently, suggestions for improvement, like imposing additional constraints or using Bayesian approaches, are examined. Chapter 3 gives a critical literature overview regarding ambiguity aversion in the context of portfolio optimization. In Chapter 4, the ambiguity-averse approach introduced by Garlappi, Uppal and Wang, which takes ambiguity aversion as well as parameter and model uncertainty into account, is presented. Chapter 5 describes the empirical applications and illustrates the out-of-sample performance of different portfolio optimization strategies in an empirical study by first replicating the original results of Garlappi, Uppal and Wang and then changing important framework conditions. The conclusion is presented in Chapter 6. Further illustrations are collected in the Appendix.

## 2. Portfolio choice

### 2.1. Modern portfolio theory

More than 60 years ago, Harry Markowitz established a whole new concept of portfolio optimization by focusing on a holistic approach of several assets to build a portfolio rather than to restrict the investment to a single seemingly profitable asset. In his article "Portfolio selection" of 1952, he derived a normative decision rule to build efficient portfolios. With this decision rule, he laid the foundation for a theory, which was later called modern portfolio theory. About 40 years later, in 1990, Harry Markowitz won the Nobel prize together with Merton H. Miller and William F. Sharpe for their pioneering work in the theory of financial economics[3].

According to Markowitz an investor should focus on two main elements when selecting the optimal portfolio, namely portfolio return and portfolio risk, where risk is defined as fluctuations in returns. He argues that each investor has different interests when choosing a portfolio, but every investor pursues the goal to achieve high returns at low risk. "It seemed obvious that investors are concerned with risk and return, and that these should be measured for the portfolio as a whole" (Markowitz, 1991, p. 470). Modern portfolio theory is based on the assumption that investors are only willing to hold a risky asset if they are compensated with a risk premium in addition to the return on the risk-free asset. Therefore, Markowitz implicitly assumes that investors are risk averse.

To determine the portfolio risk, it is important to analyze the correlations between the assets. If assets are not perfectly correlated with each other, it is possible to reduce the portfolio risk through diversification, without reducing the expected return. As a result, the portfolio risk is always below the weighted average of the individual assets' risk. As the number of assets in the portfolio increases, the role of the covariances becomes more important compared to the role of the variances of the single assets. The direct implication of this concept for the optimal portfolio is that "it is necessary to avoid investing in securities with high covariances among themselves" (Markowitz, 1952, p. 89).

Modern portfolio theory is based on numerous assumptions about the investor and the market. Besides the key assumption that investors seek to maximize yields while minimizing risk, Markowitz assumes that markets are efficient, that investors are rational and expected utility maximizers according to Bernoulli and that they make their decisions based on an individual risk function. Furthermore, the planning horizon in modern portfolio theory covers only one period. At the end of each period the investor has to make a new decision about the allocation of his capital. Asset allocation according to Markowitz is therefore a static decision with a short-term character.

In his article of 1952, Markowitz distinguishes two major steps to determine the optimal portfolio. The first step is to form expectations about future asset moments and the second step is to specify the optimal portfolio. His model is limited to the second step, which can further be divided into identifying the set of efficient portfolios and identifying the investor-specific optimal portfolio using the concept of Bernoulli expected utility maximization. From the expectations about future asset moments, the set of efficient portfolios is built. According to modern portfolio theory, a portfolio is efficient if there is no other portfolio with a higher expected return given a specific level of risk or with a lower risk given a specific level of expected return. All portfolios that are efficient lie on the efficient frontier, which represents the relationship between risk and return. In an efficient portfolio, assets with high expected returns, small variances and low correlation coefficients among each other are the ones with the highest weights. After determining the set of efficient portfolios, the next step is to select the investor-specific optimal portfolio depending on his degree of risk aversion. The optimal portfolio can be derived as the point of tangency between the efficient frontier and the highest possible Bernoulli utility curve. According to Markowitz, the optimization problem given N risky assets can be written as:

$$\max_{w} w^T \mu - \frac{\gamma}{2} w^T \Sigma w \tag{1}$$

in which $w$ is the vector of portfolio weights, $\mu$ is the vector of returns, $\gamma$ is the investor-specific degree of risk aversion and $\Sigma$ is the variance-covariance matrix.

Markowitz, however, only analyzed how to identify the optimal portfolio given the information about future asset moments, but not how these input parameters can be determined. Since information about future asset moments is unknown, the input parameters need to be estimated. Expected asset moments are typically estimated from realized returns and therefore, they are inevitably estimated with error. As opposed to that, in modern portfolio theory it is assumed that expected returns are known and as a consequence, statistical estimations are assumed to be the true values (Kalymon, 1971). Hodges and Brealey (1973) and Jorion (1985), however, show in empirical studies that historical returns only have a small forecasting power for future returns. Additionally, Kempf et al. (2002) expose in their simulation study that the estimated expected returns differ significantly from the "true" values. They also show that there is a huge gap between the optimal portfolio weights using the "true" parameters and using their estimations from historical returns. Therefore, they argue, that a rational investor should take uncertainty about estimated asset moments into account when selecting a portfolio. Barry (1974) shows that if uncertainty about future asset moments is integrated to the model, the portfolio risk can be reduced while leaving the portfolio return unchanged, leading to increasingly efficient portfolios. Chopra et al. (1993) argue that the estimation of variances and covariances can be performed more precisely than the estimation of expected returns but at the same time, estimation errors in returns have a much higher quantitative influence on the portfolio weights and the resulting portfolio performance compared to estimation errors regarding variances and covariances. Therefore, they conclude that the focus of the estimation should lie on expected returns.

One of the main drawbacks of modern portfolio theory is, that the resulting optimal portfolios are often characterized by instable portfolio weights and by extreme, non-intuitive positions (Black and Litterman, 1992). A small increase in the expected return of only one asset already leads to the reallocation of half of all assets in the portfolio, while the portfolio risk and return hardly change (Best and Grauer, 1991).

[3] For further information see Nobelprize.org

This result is not consistent with the investors' preference to allocate capital to stable portfolios that don't require frequent reallocations, since reallocations are associated with high transaction costs. Furthermore, using mean-variance optimization, the capital is allocated only to a few assets, leading to undiversified portfolios (Black and Litterman, 1992, Green and Hollifield, 1992, Broadie, 1993). These extreme portfolio weights conflict with the desire of diversification. Finally, Jobson and Korkie (1980) and DeMiguel et al. (2009) show that the optimal portfolio selected, using mean-variance optimization, can be even beaten by a uniformly diversified portfolio regarding the Sharpe ratio.

Michaud (1989) argues that the classical mean-variance approach puts high weights on assets with high expected returns, small variances and low covariances among each other, but at the same time, assets with these characteristics are often afflicted with the highest estimation errors. Therefore, assets with highly overestimated returns and with highly underestimated variances are overweighted in the optimal portfolio. This leads to an increase of the estimation error through optimization. For this reason, Michaud among others states that mean-variance portfolio optimization is "error maximization". To solve this problem, Michaud suggests an alternative, statistical, approach to estimate future asset moments. Michaud argues that the observable historical data represents only one realization of the data-generating process. Therefore, he proposes to reestimate the return distribution using Monte Carlo Simulation. From each simulated return distribution, a new set of input parameters for the subsequent optimization arises, leading to resampled efficient portfolios. Scherer (2003), however, criticizes the approach suggested by Michaud because it does not solve the cause of the bad out-of-sample performance of mean-variance portfolios induced by estimation errors regarding expected asset moments. Each portfolio, which is constructed during the simulation, is derived using the same input data. As a result, every resampled efficient portfolio has a similar deviation from the "true optimal portfolio" (Scherer, 2003). Furthermore, Fletcher and Hillier (2001) compared the performance of resampled efficient portfolios with the performance of classical mean-variance portfolios and they don't find any outperformance of either strategy.

Since the bad out-of-sample performance of mean-variance portfolios arises from estimation errors associated with expected asset moments (Merton, 1980), alternative approaches that are explicitly designed to reduce estimation errors, were established. In the following Chapters, different suggestions for improvement, like setting up additional constraints or adjusting the input parameters by the means of Bayesian approaches are analyzed and compared.

### 2.2. Implementation of additional constraints

Since different out-of-sample studies have shown that portfolios that are optimized using the classical mean-variance approach tend to be very undiversified, additional constraints can be introduced to force a higher diversification. Examples of constraints include the restriction of short selling, upper bound constraints on the position in a single asset or upper bound constraints on the exposure given to a certain country or industry (Brandt, 2009). Often, these restrictions are necessary because of legal requirements and therefore, constraints are a realistic assumption. Introducing additional constraints limits the set of possible portfolio combinations, leading to an efficient frontier that always lies below the non-restricted efficient frontier. As a result, adding constraints can never improve the ex-ante portfolio performance (Grauer and Shen, 2000). However, it has been shown that the out-of-sample portfolio performance can be improved. Restricting portfolios has a smoothening effect on portfolio weights, leading to a higher degree of diversification and to less extreme asset positions (Grauer and Shen, 2000). In a simulation study, Frost and Savarino (1988) show that introducing short selling constraints and upper bound constraints on assets improves the portfolio performance. They measure performance using certainty equivalent returns, which is the difference in portfolio utilities, if the portfolio weights are based on estimated returns and if they are based on the true parameters. Frost and Savarino conclude that the implementation of constraints is reasonable if estimated asset moments are afflicted with estimation errors. This is because the use of constraints prevents the inappropriate emphasis of estimation errors in portfolio optimization (Chopra and Ziemba, 1993).

If estimation errors, however, are not the cause of badly diversified portfolios, constraints are counterproductive, since they predetermine the optimal portfolio to a large extent ex-ante. This fact is criticized by Banz, as he states that constraints are implemented "to force the results which one desires" (Banz, 1997, p. 398). Clasing and Rudd (1988) also argue that implementing constraints is counterproductive. They state that adding restrictions implies that the investor doesn't trust the input parameters. Therefore, they conclude that it is more intuitive to directly adjust the input parameters instead of implicitly changing them by implementing constraints. Grinold and Easton (1998) agree and argue that implementing constraints doesn't treat the cause of the bad out-of-sample performance of mean-variance optimized portfolios.

Out-of-sample studies have shown that the portfolio performance can be improved by implementing constraints, but the performance is still far from good. The reason for this result is that estimated asset moments are afflicted with estimation errors that are so high that the adjustment of the optimal portfolio by adding constraints is not sufficient to solve this problem (Brandt, 2009). To overcome the counterintuitive approach of restricting the outcome by implementing constraints, new techniques have been introduced that directly adjust the input parameters. One important class of these approaches is the class of Bayesian approaches, presented in the next Chapter.

### 2.3. Bayesian approaches

#### 2.3.1. statistics

The founder of Bayesian statistics is Thomas Bayes (1702-1761). The Bayes' theorem, that was developed by him, is a fundamental theorem in the theory of probability. Bayes' rule describes the probability of an event based on prior knowledge, where $P(A)$ is the prior, the initial degree of belief in $A$ and $P(A|B)$ is the posterior, the degree of belief in $A$ after accounting for $B$. Bayes' rule can be written as follows:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \tag{2}$$

In the course of portfolio optimization, the basic idea of using Bayes' rule is, to combine a priori available information with the data to determine the posterior distribution of returns. These returns are then subsequently used for portfolio optimization. A priori information may result from financial research, news, events, macroeconomic analysis or asset pricing theories (Avramov and Zhou, 2010). A basic assumption when using Bayesian approaches in portfolio optimization is that returns are independent and identically distributed variables[4]. In the context of portfolio optimization, Bayes' rule can be written as follows (Gelman et al., 1995):

$$P(\theta|z) = \frac{P(z|\theta)P(\theta)}{\int P(\theta)P(z|\theta)d\theta} = \frac{P(z|\theta)P(\theta)}{P(z)} \tag{3}$$

in which $z = (z_1, z_2, z_3, \ldots, z_t)$ is a vector of realized returns and $\theta$ represents the unknown parameters of the probability distribution. The value of $\theta$ shall be estimated by the means of Bayesian statistics. Bayesian statistics considers $\theta$ a random variable. Information that is known ex ante about $\theta$ is combined in the a priori probability distribution $P(\theta)$. This a priori distribution is then combined with the information from the historical sample to form the a posteriori distribution $P(\theta|z)$. In contrast to methods of classical statistics where it is assumed that $\theta = th\hat{e}ta$ and therefore a point estimate of $\theta$ is provided and any potentially relevant prior information is disregarded, Bayesian statistics provides a whole distribution of $\theta$. As a consequence, it is possible to directly quantify the uncertainty about $\theta$ (Gelman et al., 1995). To solve portfolio optimization problems, it is necessary to estimate future asset returns $z$. This is done by the forecasting distribution $P(\hat{z}|z)$, in which $\hat{z}$ is the estimated value of $z$. The forecasting distribution results from the likelihood function of a future observation $P(\hat{z}|\theta)$ and the a posteriori distribution $P(\theta|z)$. Using Bayesian statistics, the distribution of future expected returns is derived by integrating over all possible values of $\theta$ weighted with their respective a posteriori probabilities (Gelman et al., 1995):

$$P(\hat{z}|z) = \int P(\hat{z}, \theta|z)d\theta = \int P(\hat{z}|\theta)P(\theta|z)d\theta \tag{4}$$

[4] See for example Jorion (1986), Black and Litterman (1992) and Pástor (2000).

Already during the 1970s, the idea to use Bayesian statistics to account for estimation errors in portfolio optimization was developed by Barry (1974) and Klein and Bawa (1976). Unlike implementing constraints, Bayesian approaches exhibit a decision-theoretical fundament. If a Bayesian approach is applied, the adjustment of the optimal portfolio takes place by directly changing the input parameters. In this way, the problem of estimation errors is directly addressed. The adjusted input parameters are then used for portfolio optimization. Barry and Klein and Bawa assume a non-informative diffuse a priori distribution, which means that they assume that all values for the future returns are considered equal probable a priori. As estimation errors are integrated in the Bayesian framework, assets are riskier since parameter uncertainty is an additional source of risk (Avramov and Zhou, 2010). As a result, in the studies of Barry and Klein and Bawa, the vector of expected returns remains unchanged but the variance-covariance matrix is multiplied by $1 + 1/n$, in which $n$ is the sample size. Consequently, if a non-informative prior is used, the set of efficient portfolios doesn't change, but the investor chooses a portfolio further to the left on the efficient frontier compared to the optimal portfolio under the classical mean-variance approach.

Brown (1976) concludes that using Bayesian approaches leads to quantifiable different portfolio structures. He adds that the results, however, depend strongly on the choice of the prior. If a non-informative prior is used, as done by Barry and Klein and Bawa, the only difference to the mean-variance approach is the multiplication of the variance-covariance matrix with the factor $1 + 1/n$ and therefore, the effect on the optimal portfolio is very small if the sample size $n$ is large. If $n$ is very large, the effect is so small that the optimal portfolio is almost identical to the optimal portfolio under the classical mean-variance approach. Avramov and Zhou (2010) state that, for all practical sample sizes, the effect of diffuse priors is negligible and therefore, in order to exploit the decisive advantage of the Bayesian approaches, it is necessary to gain informative priors for example from events, macro conditions or asset pricing theories.

#### 2.3.2. Shrinkage estimators

Shrinkage estimators go back to Charles Stein et al. (1956). Stein points out that the sample mean is not an optimal estimator for the mean of a multivariate normal distributed random variable. He argues, that if expected asset returns are estimated from the sample mean, the value is based solely on the return history of the particular asset and other potential information from the return histories of other assets are disregarded. To overcome this weakness, Stein developed an estimator that includes the return histories of all assets to determine expected returns, leading to more precise results. Stein shrinks the sample mean of each asset towards the overall sample mean of all assets, the "grand mean". This approach has a smoothening effect and prevents extreme values because expected returns of assets with comparably high or low past returns are strongest shrunk towards the grand mean. The key element of the Stein estimator is the shrinkage factor, which determines how strong the sample means are shrunk towards the grand mean. The general form of the Stein estimator (James and Stein, 1961) can be written as follows:

$$\bar{r}_j(\phi) = \phi r_0 + (1-\phi)\bar{r}_j \tag{5}$$

in which $\phi$ is the shrinkage factor, $r_0$ is the shrinkage target, $\bar{r}_j$ is the sample mean of asset $j$ and $\bar{r}_j(\phi$ is the adjusted weighted mean of asset $j$. The shrinkage factor increases in the number of assets $N$, decreases in the length of the sample size $n$ and decreases in the distance between the sample mean and the shrinkage target. Shrinkage estimators are, for example, used by Jobson (1979), Jorion (1986), Frost and Savarino (1986) and Chopra et al. (1993). Chopra, Hensel and Turner suggest a global index as shrinkage target and show that a weighted average of the individual asset mean of each country and the global index is a superior estimator of the true asset mean for each country. Frost and Savarino specify an informative prior and assume that all assets have a priori identical expected means and variances and that the pairwise correlation coefficient between any two assets is the same. They shrink the estimates of each assets' expected return, variance and pairwise correlation coefficient, determined from the historical sample, towards the average return, average variance and average correlation coefficient of all assets within the population. Therefore, the approach leads to a shrinkage towards the equal weighted portfolio and the degree of shrinkage depends on the degree to which the sample is consistent with the shrinkage target. Frost and Savarino argue that parameters with large discrepancies between the sample estimate and the overall mean of that parameter are more likely to contain estimation errors and those values are adjusted strongest towards the grand mean, reducing the estimation error. The authors show that the portfolio performance can be improved by applying their informative prior compared to applying an uninformative prior or the classical mean-variance approach.

Jagannathan and Ma (2003) point out that certain constraints that were discussed in Chapter 2.2 can be interpreted in a similar way as shrinkage estimators. They show that no short sales constraints are equivalent to reducing the estimated covariances and that upper bound constraints are equivalent to increasing the respective estimated covariances. Although the effect of both methods can be very similar, the motivation is different. While imposing constraints restricts the outcome exogenously, the adjustment of the input parameters when using shrinkage estimators is endogenous.

### 2.3.3. Bayes-Stein estimator

Jorion (1985) introduced the Bayes-Stein estimator, a further development of the classical Stein estimator. In contrast to Frost and Savarino, he assumes that estimation errors regarding the variance-covariance matrix are negligible and that variances can be estimated directly from the historical sample. Jorion argues that the minimum-variance portfolio is the only portfolio on the efficient frontier that is free of estimation errors since the determination of the minimum-variance portfolio depends solely on the variance-covariance matrix, which is assumed to be known. The more the investor moves to the right on the efficient frontier, the larger is the influence of estimation errors regarding expected returns. Therefore, the idea of Jorion is to shrink the mean-variance portfolio towards the minimum-variance portfolio, since it is robust to uncertainty about expected means and at the same time it is associated with the least risk. Using the Bayes-Stein shrinkage estimator, the sample means of the individual assets are shrunk towards the mean of the minimum-variance portfolio. The intensity of the shrinkage depends on the sample size n and on the precision of the sample mean $\nu$. The shrinkage factor $\phi$ can be computed as follows (Jorion, 1986):

$$\phi = \frac{\nu}{n+\nu} \tag{6}$$

$$\text{with } \nu = \frac{N+2}{(\bar{r}-r_0\mathbf{1})^T\Sigma^{-1}(\bar{r}-r_0\mathbf{1})} \tag{7}$$

in which $N$ is the number of assets, $r_0$ is the shrinkage target and $\bar{r}$ is the sample mean. The shrinkage factor $\phi$ is then inserted into (5) to obtain the weighted average mean for the individual asset. Since the values for $\phi$ and $r_0$ are determined directly from the sample, this approach is also called an empirical Bayes approach.

If the Bayes-Stein estimator is used to determine the optimal portfolio, the efficient frontier and the portfolio weights of the efficient portfolios don't change, but the investor will choose a portfolio more to the left on the efficient frontier, closer to the minimum-variance portfolio (Jorion, 1986). Jorion concludes that using the Bayes-Stein shrinkage estimator leads to a significantly improved out-of-sample performance. Chopra et al. (1993) confirm this result.

## 2.4. Model Uncertainty

The approaches, that have been introduced in Chapter 2.3.2 and 2.3.3, dealt with the adjustment of the sample asset moments by shrinking them towards a grand mean, by the means of Bayesian statistics. All approaches presented in the previous Chapters have in common, that solely the historical sample is used as input data to determine expected asset moments. Findings from asset pricing theory haven't been taken into account so far. In this Chapter, an alternative approach to estimate expected asset moments based on factor models is presented. Chan et al. (1999) show that using factor models to determine the input parameters for portfolio optimization significantly improves the performance of the resulting portfolios. They add, that the resulting portfolios also show a higher degree of diversification compared to a solely data-driven approach.

Pástor (2000) and Pástor and Stambaugh (2000) argue that if expected asset moments are determined exclusively from historical returns, potential information from asset-pricing models are ignored. They developed the data and model approach, in which they combine information from historical data with implications from the CAPM by the means of Bayesian statistics. They define a distinction of cases, with $\omega = 0$ meaning the investor does not believe in the return-generating model and $\omega = 1$ meaning the investor believes dogmatically in the CAPM, where $\omega$ can take values between 0 and 1. In the data and model approach, the sample means are shrunk towards the market portfolio and the degree of shrinkage depends on the shrinkage factor. The shrinkage factor measures the importance that is assigned to the CAPM and depends on the investors beliefs in the validity of the CAPM $\omega$ and on how well the historical returns can be explained by the CAPM (Pástor, 2000). Pastor and Pastor and Stambaugh argue that the investor's uncertainty about the models pricing ability can be represented by an informative a priori distribution of Jensen's alpha. If the a priori distribution of Jensen's alpha is located around zero, there is only a low degree of uncertainty and the optimal portfolio will be close to the market portfolio. If the a priori distribution of Jensen's alpha, however, is diffuse, the investor is highly uncertain about the models pricing ability and the optimal portfolio is mainly based on realized returns.

When comparing different factor models, Chan et al. (1999) show that no clear favorite specification emerges. A simple single factor model like the CAPM can't be essentially outperformed by a complex multi-dimensional k-factor model. Although the CAPM performs quite well in the study of Chen et. al., it can't capture all of the covariation among assets which could result in a systematically biased estimate of asset moments (Brandt, 2009). Consequently, the difficulty of implementing a factor model in practice is the choice of factors, since there are huge disagreements among researchers about the predictability of returns. Financial economists have identified several economic variables that predict future asset returns. However, there are large disagreements about the "true" predictive regression specification. These disagreements lead to a major source of uncertainty – model uncertainty (Avramov and Zhou, 2010). If the estimation of asset moments is based on assumptions that don't correspond with reality, estimation errors occur because of model uncertainty. The focus on one single model fails to include model uncertainty and therefore the overall uncertainty is underestimated (Cremers, 2002). If posterior odds ratios are used to rank several models and only the best model is used, it is implicitly assumed that the chosen model is true with a unit probability and model uncertainty isn't considered. Subsequently, insights from all other models are completely ignored resulting in a loss of important information (Brandt, 2009). Pástor (2000) adds that in general a model will neither be a complete reflection of reality nor will it be completely useless.

In Chapter 2.3, Bayesian approaches were introduced that incorporate parameter uncertainty by generating a weighted average of the data and the prior. This approach will now be extended to incorporate model uncertainty to portfolio optimization by implementing a weighted average of the competing return-generating models. Taking model uncertainty into account means that not only parameters, but also models, are implemented using probability distributions (Hoeting et al., 1999). The Bayesian model averaging approach (BMA), presented by Hoeting et al. (1999) determines posteriori probabilities to a set of competing models and then these probabilities are used as weights for the respective model to form an overall composite model. This overall model is then used to solve portfolio choice problems. The weights depend on the ability of the model to fit the data and on prior beliefs in the model (Hoeting et al., 1999; Raftery et al., 1997). In this context, a Bayesian approach is preferred because it allows to directly incorporate model uncertainty and is robust to model misspecification (Avramov and Zhou, 2010). Model averaging leads to higher estimates of variances compared to approaches that don't account for model uncertainty. This is why ignoring model uncertainty leads to overconfident decisions (Nigmatullin, 2003). Avramov (2002) and Cremers (2002) each introduce a Bayesian model averaging approach which integrates out both, parameter and model uncertainty. In this context, Avramov states that investors who ignore model uncertainty suffer from significant utility losses. He shows for six portfolios over the time period 1953 to 1998, that the composite model outperforms the individual model with the highest posterior probability for any model selection criterion used. Cremers stresses the high importance of taking model uncertainty into account and shows that BMA provides an improved forecasting ability compared to model selection approaches. Anderson and Cheng (2016) develop a BMA approach in which the investor updates the probabilities of each model after every period using Bayes' rule. They conclude that accounting for model uncertainty improves the out-of-sample portfolio performance.

## 3. Ambiguity aversion

### 3.1. Concept of ambiguity aversion

#### 3.1.1. A definition of ambiguity

It is well known that agents are risk averse and are willing to sacrifice expected return in order to reduce risk. But what is meant by risk? Knight (1921) was the first to note that not all sources of uncertainty can be quantified probabilistically and therefore risk is radically different from uncertainty. Knight distinguishes between two different types of uncertainty. Risk is referred to situations in which the probabilities of the outcomes are known, whereas ambiguity is referred to situations in which the probability distribution is unknown. Ambiguity might occur because the decision maker is unable or unwilling to summarize the available information by a unique probability distribution, due to a lack of information.

#### 3.1.2. The Ellsberg-experiment

While Knight was the first to differentiate between risk and ambiguity, Ellsberg (1961) was the first to show in his experiments that decision makers behave ambiguity averse.

In his first two-color urn-experiment there are two urns with 100 balls each. Urn1 contains exactly 50 red balls and 50 black balls while Urn2 also contains 100 balls, but the distribution of red balls and black balls is unknown and all combinations, ranging from 100 red balls and 0 black balls to 0 red balls and 100 black balls, are possible. Urn1 is a risky urn since the probability distribution is known. Urn2, however, is an ambiguous urn because the decision maker is unable to form a unique prior distribution over Urn2 with certainty, due to a lack of information. Decision makers were then asked whether they preferred (i) to bet on a red ball from either Urn1 or Urn2 and whether they preferred (ii) to bet on a black ball from either Urn1 or Urn2. In Ellsberg's experiment, most decision makers preferred a bet on a red ball from Urn1 to a bet on a red ball from Urn2, and a bet on a black ball from Urn1 to a bet on a black ball from Urn2. These results indicate that decision makers act ambiguity averse, in the sense that they prefer situations with known probabilities to situations with unknown probabilities.

In the second three-color urn-experiment, there is one urn that contains 90 balls. There are exactly 30 red balls in the urn and 60 balls that are either black or yellow and the distribution of black and yellow balls is unknown. The decision maker has to choose between a bet on (iii) a red ball and a bet on a black ball and between a bet on (iv) a ball that is either red or yellow and a bet on a ball that is either black or yellow. In this experiment, most decision makers preferred to bet on a red ball in situation (iii) and to bet on a ball that is either black or yellow in situation (iv). This behavior confirms the result that decision makers are ambiguity averse.

While Ellsberg's urn-experiments show the relevance of ambiguity in decision making, ambiguity averse behavior violates axioms that are necessary to derive the decision maker's subjective expected utility (Ellsberg, 1961). An agent who prefers a bet on a red ball from Urn1 to a bet on a red ball from Urn2 (i) indicates that he believes that a red ball from Urn1 is more probable than a red ball from Urn2. If the same agent prefers a bet on a black ball from Urn1 to a bet on a black ball from Urn2 (ii), this choice indicates that he believes that a black ball from Urn1 is more probable than a black ball from Urn2, which corresponds to the belief that a red ball from Urn1 is less probable than a red ball from Urn2. These two beliefs apparently contradict each other. The typical response from the second three-color urn-experiment directly violates Savage's sure-thing principle (Savage, 1954). The sure-thing principle requires that rankings of acts are independent of common parts. In the Ellsberg-experiment, situation (iii) is extended by the yellow ball in situation (iv), which is the common part. Since adding the yellow ball shifts the preference from a bet on the red ball towards a bet on a ball which is either black or yellow, this behavior violates Savage's subjective expected utility theory.

#### 3.1.3. Ambiguity aversion in the stock market

Ellsberg has shown that agents behave ambiguity averse in an experimental urn setting. Bossaerts et al. (2010) show that ambiguity aversion also plays an important role in asset markets. Investing in the stock market involves risk and ambiguity, risk whether the price will rise or fall and ambiguity about the probability distribution of outcomes. Bossaerts et al. (2010) show that ambiguity aversion has an influence on the size and the structure of the optimal portfolio holdings and also on asset prices. They show that if ambiguity is present in competitive markets, an ambiguity averse decision maker wants to hedge against this ambiguity. This desire leads to increasingly diversified portfolios as ambiguity increases. In the study of Bossaerts et. al., a significant fraction of decision makers is sufficiently ambiguity averse so that they even refuse to hold an ambiguous portfolio. Koziol et al. (2011) show that private, but also institutional investors are ambiguity averse and as a result, they reduce their allocations to risky and ambiguous portfolios. In their empirical study, the allocation to ambiguous assets is rather small and significantly below the theoretical optimal allocation. Antoniou et al. (2015) add that the willingness to invest decreases as ambiguity in the stock market increases. In their empirical study, they use dispersion in analysts' forecasts about future returns as a measure of ambiguity and equity fund flows as a measure of market participation. If ambiguity in the stock market increases, market participation strongly decreases. These results indicate that ambiguity aversion doesn't only play a role in an artificial urn environment but also in the everyday investment decisions of private and institutional investors.

### 3.2. Ambiguity averse behavior under the subjective expected utility theory

Expected utility theory assumes that the probabilities of outcomes are known and that decision makers' preferences can be presented by an individual utility function. The utility can then be computed as the expected utility of the possible outcomes weighted by their probabilities (Von Neumann and Morgenstern, 1945). In contrast, the subjective expected utility theory (SEU) introduced by Savage (1954) recognizes, that probabilities may not be objectively known. In this case, the decision maker forms his own subjective probabilities that are used for decision making. Subjective probabilities go back to Ramsey (1931), who argues that an agent's beliefs can be measured by his willingness to bet. With SEU, Savage introduced the axiomatic derivation of a Bayesian decision problem, in which probabilities and utilities are not objectively given. In the Bayesian paradigm, any source of uncertainty can be quantified probabilistically. This means, that in order to satisfy the set of axioms, even if the probabilities are not objectively known, it is necessary to compute expectations with respect to the return distribution. The Bayesian quantification of uncertainty implies that all uncertainties can be reduced to risks[5] and therefore the decision maker has a unique prior probability. As a result, the investor is only exposed to risk and trades off this risk against expected returns to maximize his expected utility (Aït-Sahali and Brandt, 2001).

Knight and Ellsberg criticize the Bayesian approach and argue that decision makers might not have enough information to form subjective expectations about the probability distribution. In this case, uncertainty can't be quantified probabilistically, which is exactly the definition of Knightian uncertainty. They argue that if investors are confronted with Knightian uncertainty, they might not be able to form a unique prior, leading to ambiguity averse behavior, as has been shown in the Ellsberg-experiment. Furthermore, the Bayesian approach does not differentiate between risk and ambiguity and therefore subjective probabilities are treated in the same way as certain probabilities. Consequently, the Bayesian approach does not represent the confidence the decision maker has in his own probabilistic assessments (Epstein and Wang, 1994). Ultimately, ambiguity averse behavior, as has been shown in the Ellsberg-experiment, cannot be modeled using Savage's Bayesian approach.

One potential path, as argued by Gilboa and Marinacci (2016), is to incorporate ambiguity aversion into the decision-making process by relaxing the assumption that decision makers are Bayesian. They state that a non-Bayesian approach doesn't require the decision maker to form a unique prior probability about expected returns and allows the decision maker to incorporate his uncertainty about the probabilities.

### 3.3. Decision-making under ambiguity

When decision makers form a belief about the probability distribution of returns, they take information like statistical models or analysis of fundamentals into account. However, the distribution of returns may remain uncertain. In this case the ambiguity averse decision maker will account for this ambiguity in his decisions.

Under the principle of insufficient reason, going back to Jakob Bernoulli, the decision maker assumes that there is no reason to expect one event to be more likely than the other and therefore one should assume that probabilities are distributed equally. Based on this assumption, the alternative is chosen which generates the highest expected utility. Gilboa and Marinacci (2016) argue that in most real-life situations there is too much information available to apply the principle of insufficient reason, but too little information to form a unique prior probability distribution as required by SEU. This dichotomy led to the development of decision rules under ambiguity, for example by taking multiple priors into account.

Many decision rules under ambiguity are based on Wald's rule (Wald, 1950). Wald's rule, which is also called minimax rule, is very conservative, as the decision is based solely on the worst possible outcome while all other outcomes are ignored. Wald's decision rule is based on the decision maker's a priori lack of confidence in his information. Using Wald's rule, the ambiguity averse decision maker assesses all acts by the minimum expected outcome associated with the respective act and then chooses the act leading to the highest minimum expected outcome (Mukerji and Tallon, 2001). Ellsberg (1961) criticizes, that if the decision is based solely on the worst possible outcome, probabilities that are known are completely ignored. He argues that in this way, the application of Wald's rule leads to a distortion of the decision maker's best estimate of probabilities towards less favorable probabilities, in which the degree of distortion depends on the level of confidence the decision maker has in his own estimated probability distribution. The less confident he is, the more he will rely on the worst possible outcome. Nevertheless, Ellsberg states that the minimax criterion might be a good starting point for a conservative decision maker in the presence of high ambiguity. Wald concludes that "a minimax solution seems, in general, to be a reasonable solution of the decision problem when an a priori distribution does not exist or is unknown" (Wald, 1950, p. 18). Applying the minimax rule can lead to a violation of Savage's sure-thing principle and is consistent with the results of the Ellsberg-experiment.

[5] Ramsey (1931) states that „For a rational man all uncertainties can be reduced to risks".

### 3.4. Incorporating ambiguity aversion into portfolio optimization

#### 3.4.1. Choquet expected utility

The Choquet expected utility model (CEU), introduced by Schmeidler (1989) was the first axiomatically sound non-Bayesian decision model for portfolio choice problems. Schmeidler's starting point was that probabilities should reflect the decision maker's willingness to bet and, under this definition, probabilities don't need to be necessarily additive. Schmeidler defines an expected utility maximization criterion for the nonadditive case, which allows the decision maker's beliefs to be presented by a unique, but not necessarily additive probability distribution. In his model, Schmeidler weakens Savage's sure-thing principle and he models probabilities by a capacity v, a set function that is not necessarily additive. If probabilities are nonadditive, the sum of the probabilities of several mutually exclusive outcomes is not necessarily equal to the probability that one of the outcomes will occur. If the sum of probabilities is smaller than the probability that one of the outcomes will occur, ambiguity aversion is reflected (Dow and da Costa Werlang, 1992). Schmeidler defines a notion of ambiguity aversion, given non-additive probabilities v as follows:

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B). \tag{8}$$

Schmeidler uses the notion of integration suggested by Choquet (1954) to generalize the case in which the capacity v is additive. Since the capacity is not necessarily additive, the weight of an outcome is determined by its place in the ranking of all possible outcomes. According to Schmeidler, a decision maker with CEU preferences is ambiguity averse if his capacity v is convex and his utility function is concave or linear.

#### 3.4.2. Maxmin expected utility

The idea of maximizing the minimum expected outcome was formalized in the model of Maxmin expected utility (MEU), introduced by Gilboa and Schmeidler (1989). The model is based on the idea that if there is not enough information available to form a single probability distribution, it is preferable to ask the decision maker to form a whole set of probability distributions $P$. Gilboa and Schmeidler argue, that if investors are uncertain about the true probability distribution, they shouldn't restrict themselves to one particular distribution as a proxy for the true probability distribution. Furthermore, considering multiple priors is advantageous, since portfolio weights react very sensitive to the choice of a certain prior and investors demand portfolios, that perform good for a whole set of possible probability distributions. The Maxmin expected utility model takes uncertainty about the true probability distribution into account by maximizing the expected utility under the worst-case scenario in the set $P$. An investor with MEU preferences evaluates each act based on the expected utility given the worst-case probability distribution in the set of prior distributions. However, the model is less conservative than Wald's rule, since not all possible prior distributions, but only a certain choice of priors is considered.

The MEU model is based on an Anscombe-Aumann framework (Anscombe et al., 1963) in which outcomes are modeled as lotteries. Gilboa and Schmeidler extend Savage's axioms by an uncertainty aversion axiom, which is the key axiom behind their model. The uncertainty axiom weakens the independence axiom of Anscombe-Aumann and states that, if the decision maker is indifferent between two uncertain acts f and g, the decision maker's preference can be written as

$$af + (1-\alpha)g \geq f(or\, g) \tag{9}$$

in which $\alpha$ is a factor between 0 and 1 (Gilboa and Schmeidler, 1989). The uncertainty axiom states that if the decision maker prefers a combination of two indifferent acts to either of the two acts, the decision maker is ambiguity averse. If the two uncertain acts $f$ and $g$ are combined, a new act arises which is not uncertain. Therefore, the two uncertain acts can be reduced to one risky act and the investor can hedge against the uncertainty by mixing the two acts (Etner et al., 2012). It can be shown that CEU is a particular case of MEU given that the decision maker is ambiguity averse. The CEU model, however, doesn't require the decision maker to be ambiguity averse and is compatible with any capacity.

#### 3.4.3. Literature overview

Following the seminal works of Schmeidler and Gilboa and Schmeidler, many further models incorporating ambiguity aversion, mostly based on CEU and/or MEU, were developed. In the following, important models are presented that give insights on how uncertainty about parameters and/or models can be incorporated to portfolio choice problems. The models introduced in this Chapter show a similar approach as the multi-prior model of Garlappi et al. (2007) presented in Chapter 4, which is used for the empirical analysis in Chapter 5.

Goldfarb and Iyengar (2003) present an optimization model under ambiguity, which is robust to parameter uncertainty. In their model, possible deviations from the expected asset moments, which are estimated from historical data, are modeled as unknown, but bounded. All parameters lie within uncertainty sets that match with confidence regions around the estimate of the parameters. The size of the uncertainty set reflects the confidence level of the decision maker. The optimal portfolio is then determined by assuming worst-case specifications of the parameters.

Tütüncü and Koenig (2004) introduce a robust asset allocation model, in which expected asset moments are modeled as uncertainty sets, which are used to obtain portfolios with the best worst-case behavior. The uncertainty sets correspond to confidence regions and are determined by bootstrapping. For expected returns, the worst-case is defined as the minimum value of the chosen confidence interval, while for the variance, the worst-case is defined as the maximum value of the chosen confidence interval. Tütüncü and König show that using this robust worst-case approach leads to a significantly better out-of-sample portfolio performance and to an increased portfolio stability compared to the mean-variance portfolio. They conclude that their approach is a reasonable alternative for conservative investors.

Wang (2005) presents a minimax approach, in which he determines the optimal portfolio numerically in a Bayesian setting, accounting for parameter and model uncertainty. In his model, uncertainty is depicted as a set of possible priors with different degrees of precision. Wang uses a shrinkage approach, in which the prior probability distribution incorporates data as well as prior beliefs generated from an asset pricing model. The resulting expected mean is a weighted average from insights of the asset pricing model and the data. Wang shows that the optimal portfolio for an investor who is ambiguity averse differs strongly from the market portfolio underlying the CAPM and also from the portfolio based solely on the sample estimate. This result indicates that model uncertainty is as important as parameter uncertainty.

Pflug and Wozabal (2007) also introduce a minimax approach, in which confidence sets are used to determine the probability distribution. The size of the confidence sets depends on the amount of information available. The more data is available, the tighter is the confidence set and the smaller is the resulting cost of ambiguity. Their approach points out the tradeoff between return, risk and model robustness.[6]

All models, introduced in this Chapter, have in common that they want to formulate optimization problems in a way, that they result in a good portfolio performance for all possible realizations of the unknown parameters. This is why these approaches are also called robust approaches. Tütüncü and Koenig (2004) and Scherer (2007) state that robust portfolio rules create portfolios which react less sensitive to new information and therefore are more robust to changes in expected returns. This is because robust decision rules lead to a shift in the portfolio from assets with high parameter uncertainty and large potential losses associated with parameter uncertainty to assets with less uncertainty, leading to an increased performance (Zhang et al., 2017). The performance of classical mean-variance estimators drops strongly even if the sample distribution differs only slightly from the assumed distribution. The performance of robust estimators reacts much weaker to deviations from the assumed distribution. However, the classical mean-variance estimator outperforms the robust estimator if the underlying distribution is correct (DeMiguel and Nogales, 2009). Chen et al. (2014) show in their empirical study that, if an ambiguity averse decision maker ignores ambiguity, welfare costs can exceed 15% compared to decision-making under robust models.

#### 3.4.4. Smooth models incorporating ambiguity aversion

The models discussed in the previous Chapter have in common that they use a minimax approach and adopt an absolute worst-case. Lim et al. (2012) criticize this approach because only the performance in the worst case is considered, ignoring the concern of underperforming in other, possibly more probable, cases. They argue that adopting an absolute worst-case approach is overly pessimistic and sensitive to the choice of the uncertainty set. In this Chapter, a smooth model incorporating ambiguity aversion is presented which is less conservative than absolute worst-case models.

Klibanoff et al. (2005) were the first to introduce a smooth model incorporating ambiguity aversion[7]. The main characteristic of their model is the separation between ambiguity, which represents the decision maker's subjective beliefs and the ambiguity attitude, which represents the decision makers tastes[8]. As a consequence of this separation, the decision maker doesn't have to select the act that maximizes the minimum expected utility, but is allowed to be less pessimistic. Klibanoff, Marinacci and Mukerji assume that the decision maker has a subjective prior probability over the set of possible probability distributions. The decision maker then computes a certainty equivalent with regard to expected utility for each probability distribution. Ultimately, the decision maker maximizes the expected utility of the conditional certainty equivalents with regard to a second utility function, which indicates the ambiguity attitude of the decision maker. Therefore, the focus is shifted from the minimum expected utility towards an aggregation of all possible expected utilities. The more ambiguity averse the decision maker is, the higher are the weights he attaches to the less favorable probability distributions. Thimme and Völkert (2015) show in their empirical study that the smooth model of Klibanoff, Marinacci and Mukerji improves the fit of the data.

The main difference to the MEU model is, that the decision maker considers various probability distributions, while Gilboa and Schmeidler only consider the worst-case distribution. Consequently, MEU is an extreme case of the smooth model when extreme ambiguity aversion is assumed (Klibanoff et al., 2005).

### 3.5. Ambiguity aversion and stock market puzzles

#### 3.5.1. Limited market participation

Ambiguity aversion plays a key role in explaining some of the most important asset pricing puzzles, like limited market participation, the equity premium puzzle or the size effect. Epstein and Schneider (2008) and Pataracchia (2011) state that ambiguity aversion affects the way how investors process information. They argue that if the quality of information is not known, investors react ambiguity averse and take a worst-case assessment of quality. In case of good news, the worst-case is that the information is noisy whereas in case of bad news, the worst-case is that the information is precise. This asymmetric reaction to information leads to a shift in mass probabilities from favorable outcomes towards less favorable outcomes (Pataracchia, 2011). In this context, several authors have shown that ambiguity aversion has an effect on market participation and can lead to a situation in which investors find it rational not to participate in the stock market at all.

Dow and da Costa Werlang (1992) state that, for a decision maker with CEU preferences, there exists a price interval of non-zero length, for which the decision maker prefers neither to buy nor to sell the risky asset and instead he strictly prefers to hold a zero position in the risky asset. They argue that this is because the decision maker acts as if he assigns different probability distributions to the long and the short position in the asset. This can lead to a situation in which the decision maker's minimum asking price for a short position in the asset, is higher than his maximum bid price for a long position in the asset. If the asset price lies somewhere in between, the decision maker does not trade at all. This behavior strongly contradicts the SEU framework, in which the decision maker buys the risky asset if the price is below his certainty equivalent and he sells the risky asset if the price is above his certainty equivalent. Under the SEU framework there is no no-trade interval. Cao et al. (2005) and Bossaerts et al. (2010) add that ambiguity aversion among investors is heterogeneous and, as a result for every ambiguity averse

[6] Further models that extend the models of Schmeidler and Gilboa and Schmeidler include Chen and Epstein (2002), Uppal and Wang (2003), Deng et al. (2005), Maccheroni et al. (2006), Ma et al. (2008) and Qu (2015). All these models have in common, that they include CEU and/or MEU as special cases.

[7] The model has been extended in Klibanoff et al. (2009) to a setting involving dynamical decision making.

[8] Ambiguity aversion is investor-specific and doesn't change over time, while ambiguity is asset-specific and does change over time.

investor, there exists an individual price region, for which he prefers not to participate in the market. The size of this region depends on the investor-specific degree of ambiguity aversion. This mechanism explains why ambiguity aversion can lead to limited stock market participation.

Epstein and Miao (2003) use this mechanism, triggered by ambiguity aversion, to explain the home bias, the limited participation in foreign markets compared to the home market. They argue that investors are more certain about the stock returns of companies in their home country and therefore they prefer to invest in the home market.

#### 3.5.2. Equity premium puzzle

The equity premium puzzle describes the disproportionate high premium for risky assets compared to the return on the risk-free asset. Although it is intuitive that there exists an equity premium because of different degrees of risk, the high amount of the equity premium is not consistent with theoretical predictions. However, most theoretical predictions take only risk into account and ignore ambiguity. Cao et al. (2005) argue that the total equity premium can be decomposed into a risk premium and an ambiguity premium. Accordingly, there exists a separate premium for bearing ambiguity additional to the premium for taking risks, explaining the high amount of the total equity premium. Izhakian (2012) defines the ambiguity premium as the amount the decision maker is willing to pay in order to replace a bet with unknown probabilities by a bet with known probabilities, given the same expected outcomes for the ambiguous and the risky bet. Pataracchia (2011) shows that asset prices decrease as ambiguity increases, because investors demand a higher ambiguity premium in order to prefer to hold the ambiguous portfolio. Anderson et al. (2009) show that excess market returns are strongly associated with ambiguity, while they are less strongly associated with risk. They conclude from their findings that the risk premium is dominated by the ambiguity premium. This result is confirmed by the empirical study of Thimme and Völkert (2015), who show that the risk premium contributes to the total equity premium by 31%, while the ambiguity premium contributes to the overall equity premium by 69%.

#### 3.5.3. Size effect

Antoniou et al. (2014) use ambiguity aversion to explain the size effect, a market anomaly, which describes that companies with a small market capitalization outperform companies with larger market capitalizations. The authors argue that analysts' forecasts about expected earnings of smaller companies are more ambiguous, since analysts find it harder to estimate the accuracy of forecasts for smaller companies, due to a lack of information. This leads to a large dispersion in analysts' forecasts about future returns. Due to the high degree of ambiguity in earnings forecasts, ambiguity averse investors will react pessimistically and overweight the worst-case scenario. Therefore, the authors show that the stock prices of small companies are reduced, compared to a situation without ambiguity aversion. At the end of the quarter, the pessimism is resolved as more information gets available and consequently ambiguity regarding expected returns decreases. This reduction of ambiguity then results in increasing stock prices. Antoniou, Galariotis and Read show empirically that this behavior leads to the size effect around earnings announcements.[9]

### 3.6. Is ambiguity aversion rational?

In Chapter 3.4, models were presented that incorporate ambiguity aversion into portfolio optimization and, in Chapter 3.5, ambiguity aversion was used to explain some of the most important puzzles in finance, indicating that ambiguity aversion does play an important role in real life investment decisions. But even if investors behave ambiguity averse, does that also mean that it is rational?

Raiffa (1961) was the first to argue that a rational decision maker should not be ambiguity averse. Scherer (2007) agrees that investors who are ambiguity averse behave irrational. He especially criticizes the MEU approach introduced by Gilboa and Schmeidler, because their model violates Savage's sure-thing principle and the model can lead to a Dutch Book outcome, a situation where the decision maker agrees to a combination of bets that guarantees him to lose. Sims (2001) agrees that violating the sure-thing principle on purpose shouldn't be recommended to decision makers. Al-Najjar and Weinstein (2009) add that admitting ambiguity averse behavior as rational can not only lead to a Dutch Book outcome, but also to situations in which the decision maker reacts sensitive to irrelevant sunk costs or is averse to information, situations that are clearly irrational. Najjar and Weinstein further criticize that in the ambiguity aversion literature the sure-thing principle is relaxed, while other aspects of Savage's framework are kept. They argue that, if ambiguity aversion is not rational but a behavioral anomaly, it doesn't seem to make sense to assume a rational decision maker in any other aspect of the model. Scherer (2007) concludes, that these drawbacks of MEU are more severe than the Ellsberg paradoxon is for subjective expected utility theory.

Al-Najjar and Weinstein (2009) give an alternative explanation for Ellsberg choices, in which the decision maker believes that the odds of the experiment are adversarial manipulated and therefore he prefers the perceived safer, risky option[10]. They argue that this behavior may be rational in real life situations, in which the counterpart might have superior knowledge about the odds, but in an experimental setting, like the Ellsberg-experiment, ambiguity averse choices are irrational and might arise from a misapplication of real life situations to the experimental setting.

[9] While Antoniou, Galariotis and Read show empirically how ambiguity aversion can create the size effect, Epstein and Schneider (2008) show theoretically, that prices under conditions with a high level of ambiguity are lower in the beginning of the quarter because investors are ambiguity averse. This pessimism about expected returns is then corrected leading to increasing prices, as soon as ambiguity starts to resolve as the quarter comes to a close.

[10] This alternative explanation of the Ellsberg paradoxon is also known as deceit aversion mechanism

A different alternative explanation for Ellsberg choices is given by Kovarik, Levin and Wang (2016), based on complexity aversion. They introduce a model in which complexity aversion, and not ambiguity aversion, is the reason why decision makers prefer the risky urn to the ambiguous urn in the Ellsberg-experiment. In their experiment, they relate complexity to the compoundness and the number of alternatives of the lotteries. They assume that decision makers want to avoid the cognitive effort related to the reduction of compound lotteries and therefore they estimate the probabilities and afterwards they discount the final value. In this way, the decision maker considers the ambiguous urn to be more complex than the risky urn. Since decision makers are assumed to be complexity averse, a higher degree of complexity reduces the utility of the lottery. As a result, a complexity averse decision maker prefers the unambiguous, less complex lottery. According to this model, Ellsberg choices don't necessarily contradict SEU, because complexity aversion affects the investor's utility and not his beliefs.

While there is some doubt about the rationality of ambiguity aversion and there exist also alternative explanations for Ellsberg choices like the deceit aversion mechanism or complexity aversion, the major part of the literature agrees that ambiguity aversion should be incorporated into portfolio choice problems. If investors care about ambiguity in the stock market and their utility depends on the degree of ambiguity, the portfolio decision rule should incorporate ambiguity. For an investor who is not ambiguity averse, however, a different, less conservative, model might be more appropriate.

### 3.7. Learning under ambiguity

Does ambiguity dissolve as the decision maker learns about his environment? If ambiguity occurs because of a lack of information about the true probability distribution, investors can learn about the missing information and ambiguity will vanish in the long run. Investors are uncertain about many parameters in financial markets and can learn about these parameters by observing data. Although huge amounts of financial data are available, the ability to learn might be limited due to the huge amount of randomness in financial markets (Pastor and Veronesi, 2009). Consequently, there might be circumstances that make it impossible for the decision maker to fully learn about the true probability distribution.

Marinacci (2002) shows that if there is repeated sampling with replacement from the same ambiguous Ellsberg urn, the decision maker learns about the true color distribution and ambiguity resolves asymptotically. However, this in only true in the special case of unambiguous signals. Several models of learning under ambiguity were established which show that ambiguity does not need to vanish completely in the long run if signals are ambiguous. Epstein and Schneider (2007) introduce a model of learning under ambiguity, which is based on the Maxmin expected utility model introduced by Gilboa and Schmeidler (1989). In their model, they distinguish between noisy signals and ambiguous signals. They argue that the draw from the ambiguous urn is an ambiguous signal, while the draw from the risky urn is a noisy signal. Since ambiguity averse decision makers presume the worst-case, the ambiguous signal is treated as if it was less precise than the noisy signal. Regarding the ambiguous urn, signals about new data are always viewed as ambiguous, also in the long run. Therefore, ambiguity about the true probability distribution may persist forever. Epstein and Schneider conclude that if ambiguous signals are considered, a learning process can be observed and ambiguity might be resolved over time, however, it doesn't vanish completely in the long run because at some point in time, the decision maker can't learn anymore about the true parameters.

Campanale (2011) introduces a model of life-cycle portfolio choice, in which investors have minimax preferences and they assume that there is a whole set of possible return distributions. Campanale shows that part of the ambiguity vanishes over time, as investors learn about the true return distribution by observing realized returns. Consequently, the set of possible return distributions shrinks, but investors are not able to identify the true return distribution with certainty. Therefore, Campanale concludes that ambiguity must also be considered in long-term investment decisions.

Branger et al. (2013) show that investors who ignore ambiguity aversion or who refuse to learn experience a utility loss. However, the loss from not learning is much smaller compared to the utility loss from ignoring ambiguity aversion[11]. The authors conclude that taking ambiguity aversion into account is of first order importance, especially since the utility loss from not learning is even reduced as ambiguity aversion increases. Binmore et al. (2012) and Guidolin and Rinaldi (2013) agree and state that learning only has a moderate effect on investment decisions.

Chapter 3 dealt with the topic of ambiguity aversion in portfolio choices. It was shown, that ambiguity aversion plays an important role in experimental settings but also in the stock market. While there is some doubt about the rationality of ambiguity aversion, and also alternative explanations were offered, the major part of the literature agrees that ambiguity aversion should be incorporated into portfolio optimization problems. Different models taking ambiguity aversion into account were presented, for example the CEU model introduced by Schmeidler, the MEU model introduced by Gilboa and Schmeidler or the smooth model incorporating ambiguity aversion introduced by Klibanoff, Marinacci and Mukerji. It was also shown that ambiguity aversion helps to explain some of the most important puzzles in finance, like limited market participation or the equity premium puzzle. The last section dealt with learning under ambiguity, however, it was argued that learning only has a moderate effect on investment decisions.

[11] Branger, Larsen and Munk show that the utility loss from refusing to learn is below 4% of the initial portfolio value even in case of a long investment horizon, while the utility loss from ignoring ambiguity aversion amounts to more than 50% of the initial wealth for an investment horizon of 20 years.

## 4. Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach (Garlappi et al., 2007)

### 4.1. Theoretical framework

This Chapter deals with the multi-prior approach introduced by Garlappi et al. (2007), that incorporates parameter and model uncertainty and takes ambiguity aversion into account. In Chapter 5, this model is then tested against other models in the literature. The model shows similarities with the models of Goldfarb and Iyengar (2003), Tütüncü and Koenig (2004) and Wang (2005), which were presented in Chapter 3.4.3. One important feature of the model is that it is able to account for parameter uncertainty and model uncertainty. Therefore, the model can be implemented if the investor estimates future returns from historical observations but also if the investor relies on a certain factor model to generate expected returns and he is uncertain whether this is the true return-generating model. Another important feature is that the model is based on a solid axiomatic foundation, since it adopts the multi-prior approach (MEU) introduced by Gilboa and Schmeidler (1989).

However, the model takes only estimation errors regarding expected returns into account, while the historical variance-covariance matrix is not adjusted. Therefore, it is assumed that $\Sigma = \hat{\Sigma}$. Although, as mentioned in Chapter 2.1, estimation errors regarding returns have a larger impact on the optimal portfolio weights than estimation errors regarding variances, it is still important to estimate variances precisely, especially because the number of covariances is in a quadratic proportion to the number of estimated returns (Ledoit and Wolf, 2003).[12] The model also does explicitly not consider learning. While there exists some evidence that ambiguity changes over time through learning, as argued in Chapter 3.7, most researchers agree that learning only has a moderate effect.

Starting from the classical mean-variance optimization model, Garlappi, Uppal and Wang (GUW) made two major adjustments to (1). Firstly, they implement an additional constraint on the expected return for each asset to lie within a confidence interval of the estimated expected return. The larger the confidence interval is, the less does the investor rely on the estimated return and as a consequence, the corresponding portfolio weight declines. If the size of the confidence interval equals 0, that is the return is estimated with perfect precision, the portfolio weights are equal to the portfolio weights of the classical mean-variance portfolio. Secondly, to account for ambiguity aversion, they implement a minimax approach, by introducing an additional minimization over the multiple priors of expected returns and/or models. The adjustments to (1) can be written as follows[13]

$$\max_{w} \min_{\hat{\mu}} w^T \hat{\mu} - \frac{\gamma}{2} w^T \Sigma w, \tag{10}$$

$$\text{subject to } f(\hat{\mu}, \Sigma) \leq \epsilon, \tag{11}$$

$$w^T \mathbf{1}_N = 1 \tag{12}$$

with (12) restricting the portfolio weights to sum to 1 and with $\epsilon$ being the ambiguity parameter, reflecting both, ambiguity and ambiguity aversion. In this model, no separation between preferences and beliefs regarding ambiguity takes place and the degree of ambiguity aversion is normalized to 1. By varying $\epsilon$, different degrees of ambiguity about the expected returns can be represented. GUW show that if the set of priors is assumed to be Gaussian, the factor $\epsilon$ directly presents the size of the confidence intervals. Function (11) depicts the additional constraint and can take different forms, depending on whether ambiguity is estimated asset by asset, jointly for all assets or for subsets of assets[14]. Which form is used depends on whether parameter uncertainty only or both, parameter and model uncertainty are considered.

If one wants to consider parameter uncertainty only, ambiguity about expected returns can be estimated jointly for all assets and the portfolio optimization problem can be solved as follows[15]

$$w_{AA}(\epsilon) = \phi_{AA}(\epsilon) w_{MIN} + [1 - \phi_{AA}(\epsilon)] w_{MV} \tag{13}$$

in which $w_{AA}(\epsilon)$ is the vector of optimal portfolio weights, $\phi_{AA}(\epsilon)$ is the shrinkage factor[16] and $w_{MIN}$ and $w_{MV}$ are the portfolio weights of the mean-variance portfolio and the minimum-variance portfolio respectively. In this special case, GUW show that the optimal portfolio weights can be written as a weighted average of the portfolio weights of the mean-variance portfolio and the minimum-variance portfolio, as is precisely the case for the Bayes-Stein shrinkage estimator. The degree of shrinkage depends on the precision of the return estimates and on the ambiguity parameter $\epsilon$. As $\epsilon$ increases, the optimal portfolio is shifted stronger towards the minimum-variance portfolio. GUW, however, show that the Bayes-Stein approach is only a special case of the ambiguity-averse model if ambiguity is estimated jointly for all assets and not in case ambiguity about expected returns is estimated separately for different subsets of assets.

[12] To account for estimation errors regarding the variance-covariance matrix, Frost and Savarino (1986) and Ledoit and Wolf (2003) use a shrinkage estimator not only for expected returns, but also for the variance-covariance matrix.

[13] See Garlappi et al. (2007, p. 48).

[14] See Garlappi et al. (2007) Proposition 1 for the optimization problem if ambiguity about expected returns is estimated asset by asset, Proposition 2 if ambiguity about expected returns is estimated for all assets jointly and Proposition 3 if ambiguity about expected returns is estimated for subsets of assets.

[15] See Garlappi et al. (2007, p. 61).

[16] For the derivation of the shrinkage factor, please see Appendix A.4.

If the investor wants to consider both, parameter and model uncertainty, the ambiguity about expected returns needs to be estimated separately for different subsets of assets. In this case, estimates about expected returns are determined by a certain factor model (e.g. the CAPM) and the investor is uncertain about the assumed return-generating model to be the true model. With regard to the model incorporating parameter and model uncertainty, GUW consider one ambiguity parameter for the risky assets and a separate one for the factors. The parameter $\epsilon_a$ represents the ambiguity about the expected asset returns, while the parameter $\epsilon_b$ represents the ambiguity about the expected returns of the factor portfolios. GUW show that if the ambiguity about the expected asset returns is relatively high, compared to the ambiguity about the expected returns of the factor portfolios, the investor prefers a portfolio that puts a higher weight on the factor portfolios compared to the risky assets and vice versa.

As argued earlier, GUW show, that if parameter uncertainty only is considered, their model shows the same structure as the Bayes-Stein approach. Therefore, Scherer (2007) argues that robust optimization models that incorporate ambiguity aversion, don't add any value to Bayesian shrinkage estimation because the set of optimal portfolios doesn't change. He states that the models don't differ from each other since the optimal portfolio weights for both models are a weighted average of the portfolio weights of the mean-variance portfolio and the minimum-variance portfolio. The only difference lies in the shrinkage factor, which again depends on the ambiguity parameter $\epsilon$. Scherer criticizes, that GUW don't state how to determine this parameter or how it is related to the risk aversion parameter $\gamma$. Consequently, the decision maker needs to form a subjective belief about the ambiguity parameter $\epsilon$. Although GUW fix the degree of ambiguity aversion to 1, the investor still needs to determine the degree of ambiguity subjectively. Furthermore, assuming a higher degree of ambiguity about expected returns has the same effect as increasing the degree of ambiguity aversion in the model of GUW, since the ambiguity parameter $\epsilon$ is defined as the product of ambiguity and ambiguity aversion.

Since the portfolio weights depend strongly on the choice of the ambiguity parameter $\epsilon$, it is important to find an objective measure for ambiguity and to separate ambiguity and attitudes toward it[17]. One example for an objective measure of ambiguity was introduced by Izhakian (2012). He suggests to measure ambiguity by the variance of probabilities with a parameter ranging from 0, given that all probabilities are perfectly known to 1 in case the probability is either 0 or 1 with equal probabilities. This measure of ambiguity makes it also possible to rank alternatives according to their degree of ambiguity.

[17] See Klibanoff et al. (2005), for a model which separates between ambiguity and ambiguity aversion. As mentioned in Chapter 3.4.4, it can be shown that separating preferences from beliefs leads to a better fit of the data, however, the parameters in this model also need to be determined subjectively.

### 4.2. Empirical results

#### 4.2.1. Empirical setting

GUW test their ambiguity-averse approach against various other models in the literature, like the classical mean-variance approach, the minimum-variance approach and the Bayes-Stein approach. As a performance measure, they use the Sharpe ratio and therefore they test whether even a decision maker with mean-variance preferences can achieve a higher utility by taking ambiguity into account. To measure the utility of an ambiguity averse investor, it is necessary to characterize ambiguity averse preferences by a utility function, in which the utility increases as ambiguity decreases. Izhakian (2012) developed an expected utility framework that incorporates ambiguity, in which expected utility increases as the expected return increases, and utility decreases as the standard deviation increases or the degree of ambiguity increases. Izhakian further develops an extended measure of the Sharpe ratio, that incorporates ambiguity. This is a better measure for an ambiguity averse investor to compare different portfolio optimization strategies, since investors are willing to accept a lower Sharpe ratio if the portfolio ambiguity decreases (Koziol et al., 2011).

GUW use returns on eight international equity indices from January 1970 to July 2001 to measure the out-of-sample performance of the different portfolio optimization strategies. The portfolio weights are determined using historical returns from a rolling window of 120 months and the risk aversion parameter $\gamma$ is set to 1. A risk aversion parameter of 1 corresponds to a low degree of risk aversion, since empirical measures have shown that the degree of risk aversion typically lies in a range between 2 and 4[18]. This low degree of risk aversion contradicts the very conservative approach of GUW and is not consistent with the extreme shrinkage towards the minimum-variance portfolio (Scherer, 2007). Scherer (2007) argues, that this contradiction results from the separation of risk preferences and ambiguity preferences, since the optimal portfolio depends strongly on the relation between risk aversion and ambiguity aversion. Therefore, it doesn't seem realistic to assume a low degree of risk aversion and at the same time a high value for the ambiguity parameter $\epsilon$. To show the effect of the choice of the risk aversion parameter $\gamma$, the empirical study of GUW is replicated using a different value for $\gamma$ in Chapter 5.3.2.

The empirical results of GUW are divided into two parts. One part considering the model accounting for parameter uncertainty only and the second part, in which the model that additionally takes model uncertainty into account is considered.

#### 4.2.2. Parameter uncertainty only

With regard to GUW's first empirical application, it is accounted for parameter uncertainty only and the ambiguity about all assets is estimated jointly. Therefore, the optimal portfolio weights can be expressed as (13). The investor can

[18] See for example Paravisini et al. (2016)

build his portfolio from eight international equity indices and a risk-free asset doesn't exist. When comparing the Sharpe ratios of the mean-variance strategy, the minimum-variance strategy, the Bayes-Stein strategy and the ambiguity-averse approach, the authors show that the portfolios incorporating ambiguity aversion have a higher mean, lower variance and therefore higher Sharpe ratio compared to the classical mean-variance portfolio. For a sufficient high degree of ambiguity ($\epsilon > 0.25$), the ambiguity-averse approach also outperforms the Bayes-Stein model. This result, however, follows from the good performance of the minimum-variance portfolio for this specific data set, which shows the highest mean, lowest variance and therefore also the highest Sharpe ratio of all portfolios. Since the ambiguity-averse portfolio is a weighted average of the mean-variance portfolio and the minimum-variance portfolio, and the portfolio is strongly shifted towards the minimum-variance portfolio for a sufficient high degree of the ambiguity parameter $\epsilon$, this explains the good performance of the ambiguity-averse approach.

In the empirical study of GUW, the minimum-variance portfolio shows the best out-of-sample performance, this, however, is not true in general. The data set GUW used for their empirical study has specific characteristic, e.g. returns are very noisy. Because returns are noisy, using this data set leads to an optimal strategy, that does not take expected returns into account at all. Therefore, these data-specific characteristics put an advantage to the minimum-variance approach. In Chapter 5.4, it is shown that for a different data set, it is not optimal to only hold the minimum-variance portfolio. This also reduces the superiority of the ambiguity-averse approach, compared to the classical mean-variance approach. Because of specific characteristics of historical data, Scherer (2007) argues that applying a rolling window analysis to a historical sample path differs from real out-of-sample testing. Therefore, it is important to vary certain assumptions to test the robustness of results based on historical data. A variation of important model assumptions is performed in Chapter 5.3 and in Chapter 5.4 the portfolio optimization problem is applied to a different data set.

The authors also show that the portfolios incorporating ambiguity aversion are more stable over time and show less extreme portfolio weights compared to the mean-variance portfolio and the Bayes-Stein portfolio. The higher the ambiguity parameter $\epsilon$, the more stable are the portfolio weights over time. This is because a high parameter $\epsilon$ implies a large confidence interval resulting in a more conservative approach. In Chapter 5, a measure for portfolio turnover is introduced to directly compare the portfolio stability of different portfolio optimization strategies.

#### 4.2.3. Parameter and model uncertainty

Regarding GUW's second empirical application, it is accounted for both, parameter and model uncertainty, and ambiguity is estimated separately for different subsets of assets. In this case, GUW assume that returns are generated by a factor model and investors are uncertain about the validity of the return-generating model. In their empirical study, GUW assume the return-generating model to be the CAPM and therefore the factor portfolio is the world market portfolio. In this model, the investor has the possibility to invest in eight international equity indices and in the factor portfolio, but again the investor is not allowed to invest in the risk-free asset. Similar to the model of Pástor (2000), discussed in Chapter 2.4, the investor's belief about the validity of the CAPM is reflected by the parameter $\omega$, where $\omega = 0$ means the investor does not believe in the factor model and estimates expected asset returns from the historical sample and $\omega = 1$ means the investor believes dogmatically in the model and he estimates expected asset returns from the factor model. The expected return of the factor portfolio is estimated from the historical sample in both cases. In this application, the investor considers ambiguity about the estimated asset returns $\epsilon_a$ and about the estimated returns of the factor portfolio $\epsilon_b$. If $\epsilon_a$ and $\epsilon_b$ are both 0, the resulting optimal portfolio is the mean-variance portfolio in case of $\omega = 0$ and the market portfolio in case of $\omega = 1$. GUW show that a small degree of ambiguity about the asset returns and the returns of the factor portfolio leads to an improved out-of-sample performance compared to the mean-variance portfolio and the Bayes-Stein portfolio. For large values of $\epsilon_a$ and $\epsilon_b$, however, the performance of the ambiguity-averse portfolio declines.

Although the model incorporates model uncertainty, the authors consider only the extreme cases where the investor doesn't believe in the model at all ($\omega = 0$) or where the investor believes dogmatically in the model ($\omega = 1$). Jacquier and Polson (2010) argue that most investors have beliefs somewhere in between. Avramov (2003) states that it is neither optimal to base a portfolio on dogmatic beliefs in a return-generating model nor to completely ignore the insights from the model. He argues that, combining both sources of information, the historical sample and the return-generating model, results in higher out-of-sample Sharpe ratios. Furthermore, in GUW's approach, the investor has to make a subjective decision about his belief in the return-generating model. Instead of subjective beliefs, the parameter $\omega$ could be determined empirically, for example using posterior odd ratios or Bayesian model averaging if more than one return-generating model is considered, as discussed in Chapter 2.4.

Chapter 4 introduced the model of portfolio selection incorporating ambiguity aversion developed by GUW. In Chapter 5, the out-of-sample performance of different portfolio optimization strategies is compared with a special focus on investors who are ambiguity averse.

## 5. Empirical applications of the ambiguity-averse approach

### 5.1. Empirical design

The empirical analysis consists of three different parts. In the first part, the empirical design of GUW is adopted and their results are replicated with focus on differences in results. Both models are analyzed, the one incorporating parameter uncertainty only and the one that takes additionally model uncertainty into account. In the first portfolio optimization problem, with parameter uncertainty only, the investor can build his portfolio from eight international equity indices (MSCI Italy, MSCI Canada, MSCI France, MSCI Germany, MSCI Japan, MSCI Switzerland, MSCI UK and MSCI USA) and a risk-free asset doesn't exist. With regard to the second portfolio optimization problem, incorporating parameter and model uncertainty, the investor additionally has the possibility to invest in the factor portfolio (MSCI World). In this second case, in order to determine the expected excess returns, the month-end US 30-day T-bill is subtracted from the expected returns. Therefore, a risk-free asset exists but it is not possible to invest in the risk-free asset. The data is computed from Datastream and from the Center for Research in Security Prices (CRSP) for the timespan from January 1970 to July 2001. The timespan is identical to the one proposed by GUW. Following the approach of GUW, a rolling window of 120 months is used to determine the optimal portfolio weights for the 121st month. These portfolio weights are then used to compute the portfolio return in the 121st month. The resulting out-of-sample period includes the timespan from January 1980 to July 2001 (259 observations). In Chapter 5.3.1, the window length is varied to test the impact of the length of the sample size on the portfolio performance and in Chapter 5.3.3 the time period is extended to December 2016.

In all empirical analyses, the models of GUW are tested against the classical mean-variance model, the minimum-variance model, the Bayes-Stein model and against the model of naïve diversification, referred to as the 1/N rule. These are the same models used by GUW, except for the 1/N rule. This model is included because of evidence that the 1/N rule can't be consistently outperformed by complex models like the one proposed by GUW (e.g. DeMiguel et al. (2009)). To compare the performance of the different portfolio optimization strategies, the Sharpe ratio and the turnover ratio are computed. The Sharpe ratio is derived as follows:

$$\text{Sharpe ratio} = \frac{\mu - r_f}{\sigma}. \tag{14}$$

The risk-free rate $r_f$ exists, however, only with regard to the second model incorporating parameter and model uncertainty for the CAPM to be valid. Therefore, the Sharpe ratio reduces to the mean-to-standard-deviation ratio with regard to the first model taking only parameter uncertainty into account. This approach, chosen by GUW, in the second case is rather inconsistent, since the investor is not allowed to invest in the risk-free asset, but the risk-free asset is still used to evaluate the performance and for the CAPM to be valid. This approach also makes a comparison of the portfolios including and excluding the market portfolio impossible, since different performance measures are used. Nevertheless, this approach is adopted to make sure that the replicated results are comparable with the original results of GUW. Furthermore, a significance test of Sharpe ratios, as suggested by Jobson and Korkie (1981) and corrected by Memmel (2003), is performed.

However, as already mentioned in Chapter 4.2.1, if an investor is ambiguity averse and his utility depends on the degree of ambiguity, a performance measure should be implemented that takes ambiguity into account. Since the Sharpe ratio only considers risk and return, it doesn't seem plausible to expect ex ante a higher Sharpe ratio for portfolios that account for ambiguity aversion. If, nevertheless, the Sharpe ratio is used as a performance measure, it is tested whether the investor can achieve a higher out-of-sample performance by taking ambiguity into account, even if he has mean-variance preferences.

In addition to the Sharpe ratio, the portfolio turnover is computed to measure the stability of the portfolio weights. In practice, investors prefer stable portfolios that don't have to be reallocated heavily every month because they take transaction costs into account. The turnover ratio is calculated as follows[19]

$$\text{Turnover} = \frac{1}{T-n}\sum_{t=1}^{T-n}\sum_{j=1}^{N}(|\hat{w}_{j,t+1} - \hat{w}_{j,t^+}|) \tag{15}$$

in which $T-n$ is the length of the out-of-sample period, with $T$ being the total length of the data series and $n$ being the length of the sample size, $N$ is the number of assets, $\hat{w}_{j,t^+}$ are the portfolio weights just before rebalancing and $\hat{w}_{j,t+1}$ are the portfolio weights right after rebalancing at time $t+1$. The turnover ratio measures the shift in portfolio weights every period and is therefore also referred to as relative turnover.

After replicating the results of GUW in Chapter 5.2, different parameters like the sample size, the degree of risk aversion and the timespan are varied in Chapter 5.3. By varying the input parameters, it is measured whether the out-of-sample performance of the different portfolio optimization strategies is robust to changes in the empirical design. Furthermore, the robustness of the superiority of the ambiguity-averse approach compared to other portfolio optimization models, as found by GUW, is tested. Finally, in Chapter 5.4, the investment universe is changed and the portfolio optimization problem is applied to the German DAX30 companies.

### 5.2. Replication of the empirical results of Garlappi, Uppal and Wang

#### 5.2.1. Parameter uncertainty only

The first part of the empirical analysis deals with the replication of the empirical results of GUW. For the first empirical application, the ambiguity-averse model incorporating parameter uncertainty only is tested against the classical mean-variance model, the minimum-variance model, the Bayes-Stein model and the model of naïve diversification. With regard to this portfolio optimization problem, the investor

[19]See DeMiguel et al. (2009, p. 1929).

has the possibility to invest in eight international equity indices and a risk-free asset doesn't exist. The derivation of the portfolio weights for each strategy can be found in Appendix A. The optimal portfolio weights for the portfolios incorporating ambiguity aversion are determined from (13). Using (13), different values of the ambiguity parameter $\epsilon$ ranging from 0 to infinity are applied. However, GUW don't give any guidance on how $\epsilon$ can be determined. Therefore, the choice of $\epsilon$ remains a subjective choice of the investor. Just as done by GUW, two cases are considered, one, where short selling is allowed (Panel A) and one, where short selling is not allowed (Panel B). The original results and the replication of the results are presented in Table 1.

The replicated results are very similar to the original results of GUW. The main difference is that the mean of the mean-variance strategy in Panel A is somewhat higher, while the mean of the minimum-variance strategy in both Panels is slightly below the one found by GUW. This reduces the superiority of the minimum-variance strategy over the mean-variance strategy, but the Sharpe ratio of the minimum-variance strategy is still considerably higher and therefore, the findings presented by GUW are still observable.

The ambiguity-averse portfolios show in both Panels a higher Sharpe ratio compared to the mean-variance portfolio and compared to the Bayes-Stein portfolio, given a degree of ambiguity of 0.5 or higher. If $\epsilon$ is about 0.25, the shrinkage factor of the ambiguity-averse portfolio is approximately equivalent to the Bayes-Stein shrinkage factor while for higher values of $\epsilon$, the shrinkage factor of the ambiguity-averse portfolio increases. In Figure 1, the shrinkage factor of the Bayes-Stein portfolio $\phi_B S$ and the shrinkage factors of two portfolios incorporating ambiguity aversion $\phi_A A$ with $\epsilon = 1$ and $\epsilon = 3$, in case that short selling is allowed, are compared[20]. From Figure 1 it can be seen that the portfolios incorporating ambiguity aversion are shrunk much more towards the minimum-variance portfolio and the shrinkage factors are more stable over time. This result is consistent with the results found by GUW. The strong shrinkage towards the minimum-variance portfolio explains the superior performance to the Bayes-Stein approach for high values of $\epsilon$.

From Table 1, however, it can be seen that the out-of-sample performance of the ambiguity-averse approach depends strongly on the degree of ambiguity $\epsilon$. It is therefore necessary to measure ambiguity objectively and to separate between ambiguity (belief) and ambiguity aversion (taste). Only ambiguity aversion should depend on the subjective preference of the decision maker. Without any guidance on how to determine $\epsilon$, it can't be shown that a portfolio optimization strategy incorporating ambiguity aversion outperforms other models, like the mean-variance model or the Bayes-Stein model, in general.

A model of naïve diversification was additionally tested. The out-of-sample Sharpe ratio of the uniformly diversified portfolio, in case short selling is allowed, is higher than the Sharpe ratio of the mean-variance portfolio and the Bayes-Stein portfolio and is only outperformed by the minimum-variance portfolio and by the portfolios incorporating ambiguity aversion for extreme high levels of ambiguity ($\epsilon > 3$). The 1/N rule, however, is outperformed by the Bayes-Stein approach and the ambiguity-averse approach in case short selling is not allowed and a sufficient high amount of ambiguity is considered ($\epsilon \geq 0.25$). If short selling is prohibited, the difference in out-of-sample Sharpe ratios decreases, since imposing a short sale constraint improves the out-of-sample performance of the mean-variance portfolio, as discussed in Chapter 2.2. At the same time, the performance of the minimum-variance portfolio changes only little since the portfolio weights are less extreme and the Sharpe ratio of the uniformly diversified portfolio doesn't change at all. Consequently, the superiority of the minimum-variance approach and the 1/N rule decreases if short selling is prohibited.

Table 2 shows the results of the Jobson-Korkie significance test of Sharpe ratios for the different portfolio optimization strategies when short selling is allowed. While the Sharpe ratios of the ambiguity-averse portfolios (with $\epsilon = 1$ and $\epsilon = 3$) are significantly higher than the Sharpe ratios of the mean-variance portfolio and the Bayes-Stein portfolio on the 5%-confidence level, they don't differ significantly from the Sharpe ratio of the minimum-variance portfolio. This is because the ambiguity-averse portfolios are shrunk strongly towards the minimum-variance portfolio and therefore the portfolio structures are very similar. The Sharpe ratio of the mean-variance strategy is not significantly different from the Sharpe ratio of the minimum-variance strategy on the 5%-confidence level, however, this is true for the 10%-confidence level.[21]

From the significance test, it can be observed that the Sharpe ratios of the ambiguity-averse portfolio with $\epsilon = 3$ and the uniformly diversified portfolio are very similar, since the value of the $z$-statistic is very low ($p$-value = 0.9656). This indicates that the performance of the simple strategy of naïve diversification does not differ significantly from the performance of a complex model incorporating ambiguity, even if a high degree of ambiguity is considered. Note, however, that the degree of ambiguity is not taken into account in the performance measure. Therefore, this conclusion is only true for an investor with mean-variance preferences. In order to test the validity of the results for an investor who is ambiguity averse, it is necessary to develop a performance measure which accounts for ambiguity.

Additionally, to the Sharpe ratio, the turnover ratio was computed as a performance measure. In both cases, whether short selling is allowed or not, the portfolio turnover is highest for the mean-variance portfolio and lowest for the uniformly diversified portfolio. Besides the uniformly diversified

[20]The derivation of the Bayes-Stein shrinkage factor can be found in Appendix A.3, while the derivation of the shrinkage factor of the ambiguity-averse portfolios can be found in Appendix A.4.

[21]However, Jobson and Korkie (1981) state that the statistical power of the Jobson-Korkie significance test is low. Thus, a $z$-statistic which indicates that the Sharpe ratios are significantly different is a strong evidence.

**Table 1:** Replication with parameter uncertainty only

This table reports the out-of-sample mean-to-standard-deviation ratio and turnover ratio for the returns on different portfolio optimization strategies. The parameter $\epsilon$ presents the ambiguity parameter and varies between 0 and infinity. The values for the means and the standard deviations are expressed as percentage per month. Turnover is expressed as shift in portfolio weights per month. The degree of risk aversion $\gamma$ is assumed to be 1.

| Strategy | Original results of GUW | | | Replication of results | | | |
|---|---|---|---|---|---|---|---|
| | Mean | SD | Mean/SD | Mean | SD | Mean/SD | Turnover |
| Panel A: Short sales allowed | | | | | | | |
| Mean-Variance | 0.0049 | 0.2557 | 0.0190 | 0.0064 | 0.2550 | 0.0252 | 2.0584 |
| Minimum-Variance | 0.0118 | 0.0419 | 0.2827 | 0.0109 | 0.0421 | 0.2589 | 0.0723 |
| Bayes-Stein | 0.0071 | 0.1058 | 0.0671 | 0.0066 | 0.1203 | 0.0552 | 0.9298 |
| Naive Diversification | - | - | - | 0.0097 | 0.0439 | 0.2214 | 0.0000 |
| Ambiguity-averse | | | | | | | |
| $\epsilon = 0.00$ | 0.0049 | 0.2557 | 0.0190 | 0.0064 | 0.2550 | 0.0252 | 2.0584 |
| $\epsilon = 0.25$ | 0.0065 | 0.1307 | 0.0495 | 0.0068 | 0.1323 | 0.0515 | 1.0343 |
| $\epsilon = 0.50$ | 0.0080 | 0.0953 | 0.0841 | 0.0080 | 0.0981 | 0.0813 | 0.7136 |
| $\epsilon = 0.75$ | 0.0091 | 0.0765 | 0.1190 | 0.0088 | 0.0796 | 0.1112 | 0.5411 |
| $\epsilon = 1.00$ | 0.0098 | 0.0654 | 0.1491 | 0.0094 | 0.0682 | 0.1375 | 0.4374 |
| $\epsilon = 1.50$ | 0.0104 | 0.0545 | 0.1909 | 0.0099 | 0.0562 | 0.1755 | 0.3208 |
| $\epsilon = 2.00$ | 0.0107 | 0.0501 | 0.2144 | 0.0101 | 0.0511 | 0.1977 | 0.2582 |
| $\epsilon = 2.50$ | 0.0109 | 0.0480 | 0.2281 | 0.0103 | 0.0487 | 0.2107 | 0.2211 |
| $\epsilon = 3.00$ | 0.0111 | 0.0467 | 0.2369 | 0.0104 | 0.0473 | 0.2189 | 0.1973 |
| $\epsilon \to \infty$ | 0.0118 | 0.0419 | 0.2827 | 0.0109 | 0.0421 | 0.2589 | 0.0723 |
| Panel B: Short sales not allowed | | | | | | | |
| Mean-Variance | 0.0104 | 0.0587 | 0.1774 | 0.0103 | 0.0587 | 0.1752 | 0.2072 |
| Minimum-Variance | 0.0117 | 0.0412 | 0.2831 | 0.0107 | 0.0415 | 0.2568 | 0.0481 |
| Bayes-Stein | 0.0106 | 0.0511 | 0.2074 | 0.0101 | 0.0435 | 0.2322 | 0.1090 |
| Naive Diversification | - | - | - | 0.0097 | 0.0439 | 0.2214 | 0.0000 |
| Ambiguity-averse | | | | | | | |
| $\epsilon = 0.00$ | 0.0104 | 0.0587 | 0.1774 | 0.0103 | 0.0587 | 0.1752 | 0.2072 |
| $\epsilon = 0.25$ | 0.0113 | 0.0511 | 0.2214 | 0.0101 | 0.0446 | 0.2270 | 0.1183 |
| $\epsilon = 0.50$ | 0.0115 | 0.0482 | 0.2391 | 0.0103 | 0.0427 | 0.2402 | 0.0906 |
| $\epsilon = 0.75$ | 0.0115 | 0.0467 | 0.2472 | 0.0104 | 0.0421 | 0.2458 | 0.0774 |
| $\epsilon = 1.00$ | 0.0116 | 0.0457 | 0.2533 | 0.0104 | 0.0419 | 0.2487 | 0.0702 |
| $\epsilon = 1.50$ | 0.0116 | 0.0446 | 0.2607 | 0.0105 | 0.0417 | 0.2516 | 0.0629 |
| $\epsilon = 2.00$ | 0.0117 | 0.0440 | 0.2647 | 0.0105 | 0.0416 | 0.2531 | 0.0592 |
| $\epsilon = 2.50$ | 0.0117 | 0.0436 | 0.2671 | 0.0106 | 0.0416 | 0.2539 | 0.0570 |
| $\epsilon = 3.00$ | 0.0117 | 0.0433 | 0.2695 | 0.0106 | 0.0416 | 0.2544 | 0.0556 |
| $\epsilon \to \infty$ | 0.0117 | 0.0412 | 0.2831 | 0.0107 | 0.0415 | 0.2568 | 0.0481 |

portfolio, which does not show any relative turnover at all[22], the minimum-variance portfolio has the lowest turnover ratio and the turnover ratios of the Bayes-Stein portfolio and the ambiguity-averse portfolios lie between the turnover ratio of the mean-variance portfolio and the minimum-variance portfolio. Since the shrinkage factors of the ambiguity-averse portfolios for a sufficient high level of $\epsilon$ are higher than the shrinkage factor of the Bayes-Stein portfolio, this explains the lower turnover of portfolios accounting for ambiguity aversion compared to the Bayes-Stein portfolio. If short selling is not allowed, the turnover of the mean-variance portfolio declines strongly, but is still considerably higher than the turnover of the minimum-variance portfolio. This reduction of turnover can again be explained by the improvement of the mean-variance strategy by imposing a short selling constraint. If short sales are prohibited, portfolio weights are less extreme and the portfolio turnover decreases. A graphical representation of the fluctuation of portfolio weights for the different portfolio optimization strategies, as was also presented by GUW, can be found in Appendix B.

[22]The uniformly diversified portfolio, however, shows absolute turnover. The portfolio weights don't change over time, but absolute prices change and therefore also the 1/N rule requires monthly reallocation (DeMiguel et al., 2009).

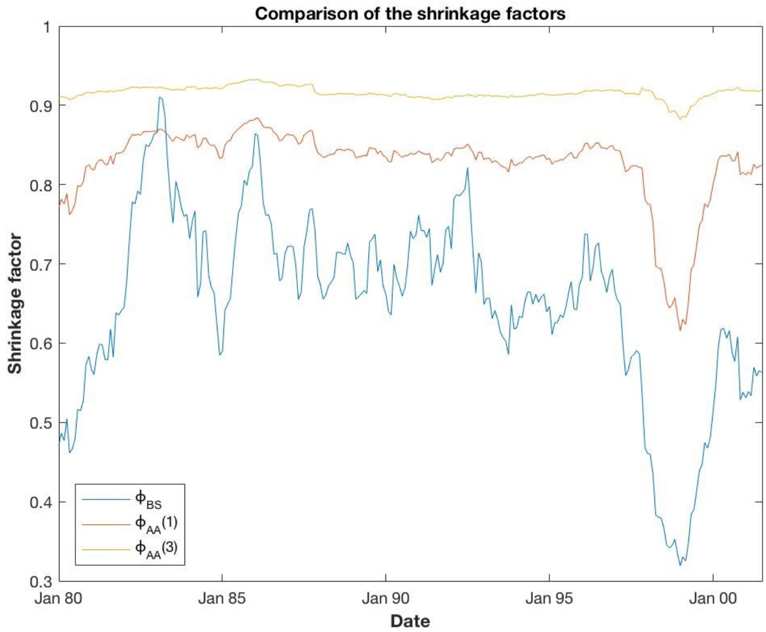



**Figure 1:** Comparison of the shrinkage factors

This figure compares the shrinkage factors of the Bayes-Stein portfolio and two portfolios incorporating ambiguity aversion. The blue line shows the shrinkage factor of the Bayes-Stein approach, the red line shows the shrinkage factor if the ambiguity parameter $\epsilon$ is set to 1 and the yellow line shows the shrinkage factor if the ambiguity parameter $\epsilon$ is set to 3. The degree of risk aversion $\gamma$ is assumed to be 1.

**Table 2:** Jobson-Korkie significance test – replication

This table reports the values of the z-statistics according to the significance test of Sharpe ratios suggested by Jobson and Korkie (1981) and corrected by Memmel (2003). The Sharpe ratios of the mean-variance strategy (MV), the minimum-variance strategy (MIN), the Bayes-Stein strategy (BS), the 1/N rule (ND), the ambiguity-averse approach with $\epsilon = 1$ ($\epsilon = 1.00$) and the ambiguity-averse approach with $\epsilon = 3$ ($\epsilon = 3.00$) are tested. One star * means that the p-value indicates that the Sharpe ratios are significantly different on the 5%-confidence level, while two stars ** indicate that the Sharpe ratios are significantly different on the 1%-confidence level.

Jobson-Korkie significance test

| Strategy | MV | MIN | BS | ND | $\epsilon = 1.00$ | $\epsilon = 3.00$ |
|---|---|---|---|---|---|---|
| Mean-Variance | 1.0000 | -1.9474 | -0.9388 | -1.6444 | -2.2130* | -2.3380* |
| Minimum-Variance | | 1.0000 | 1.8858 | -0.8964 | 1.4216 | 0.8728 |
| Bayes-Stein | | | 1.0000 | 1.5252 | -2.8397** | -2.3352* |
| Naive Diversification | | | | 1.0000 | 0.9438 | 0.0432 |
| $\epsilon = 1.00$ | | | | | 1.0000 | -1.8133 |
| $\epsilon = 3.00$ | | | | | | 1.0000 |

5.2.2. Parameter and model uncertainty

With regard to the second empirical application, the ambiguity-averse approach incorporating both, parameter and model uncertainty, is tested against the mean-variance model, the minimum-variance model, the Bayes-Stein model and the model of naïve diversification. GUW also include Pastor's data and model approach presented in Chapter 2.4, but since it is nested in the ambiguity-averse approach (in case $\epsilon_a$ and $\epsilon_b$ are both 0), this approach isn't explicitly considered here. With regard to this portfolio optimization problem, the investor has the possibility to invest in the market portfolio in addition to the eight international equity indices. For the CAPM to be valid, a risk-free asset is introduced, but the investor is not allowed to invest in the risk-free asset.

The optimal portfolio weights for the ambiguity-averse portfolios are determined from the equation system (A.5), which can be found in Appendix A.5. Using (A.5), different values for the ambiguity about the expected asset returns $\epsilon_a$ and about the expected returns of the factor portfolio $\epsilon_b$ are applied. The equation system (A.5) doesn't restrict the portfolio weights to sum to 1 and, at the same time, the investor is not allowed to invest in the risk-free asset. Therefore, in order to solve for the optimal portfolio weights, the resulting weights for the risky assets $w_a$ and the factor portfolio $w_b$ need to be upscaled to one. This procedure seems to be implausible because, if the portfolio weights are very small or very large, they are strongly modified to make sure that the investor invests his whole wealth in the stock market. Furthermore, if the ambiguity about the expected asset returns $\epsilon_a$ is very high and at the same time the ambiguity about the expected returns of the market portfolio $\epsilon_b$ is very high, the investor might not want to participate in the stock market at all. This result of limited market participation, in case that ambiguity is sufficiently high, was discussed in Chapter 3.5.1. GUW, however, don't give any guidance on how the investor should behave if the degree of ambiguity is so high that he prefers not to participate in the stock market at all.

The original results of the portfolio optimization problem with parameter and model uncertainty and the replicated results are shown in Table 3. Following the approach of GUW, the tables are divided into two parts. One, where the investor does not believe in the CAPM ($\omega = 0$) and the expected excess returns of the assets $\hat{\mu}_a$ are estimated from the historical sample, and one, where the investor believes dogmatically in the CAPM ($\omega = 1$) and $\hat{mu}_a$ is estimated from the CAPM in which $\hat{mu}_a = \bar{\beta}\hat{\mu}_b$ and $\bar{\beta}$ is the N x 1 vector of betas. The expected excess return of the market portfolio $\hat{\mu}_b$ is estimated from the historical sample in both cases.

From Table 3 it can be seen, that GUW show in their results that the out-of-sample Sharpe ratio of the mean-variance portfolio and the Bayes-Stein portfolio can be improved by allowing for a moderate level of ambiguity. However, the Sharpe ratio of the ambiguity-averse portfolio declines if $\epsilon_a$ and $\epsilon_b$ are large. If $\epsilon_a$ is relatively high compared to $\epsilon_b$, the investor prefers a portfolio which puts high weights on the world market portfolio. In the extreme case, the optimal portfolio equals the world market portfolio and the Sharpe ratio is 0.1239. If, however, $\epsilon_b$ is relatively high compared to $\epsilon_a$, the portfolio is shifted towards a portfolio consisting only of the risky assets. This effect is stronger in case the investor does not believe in the CAPM ($\omega = 0$), while the portfolio structure, in case the investor believes dogmatically in the CAPM ($\omega = 1$) depends only little on $\epsilon_a$, whereas it is strongly affected by changes in $\epsilon_b$.

The results of the replication show a different, however, intuitive picture of the out-of-sample performance of the different portfolio optimization strategies. The first difference to be noted is that the Sharpe ratios of the mean-variance portfolio and the Bayes-Stein portfolio are considerably higher compared to the original results of GUW. For the replication, when applying these two strategies, it was assumed that the investor has the possibility to invest in nine different assets, the eight international equity indices plus the world market portfolio and a separation between the risky assets and the world market portfolio didn't take place. The missing separation when applying the mean-variance strategy, also explains why the Sharpe ratio of the mean-variance portfolio differs from the Sharpe ratio of the ambiguity-averse portfolio with $\epsilon_a = 0$ and $\epsilon_b = 0$.

The main difference between the original results and the replicated results, however, is the difference in Sharpe ratios of the portfolios incorporating ambiguity aversion. If $\epsilon_b = 0$ and $\epsilon_a > 0$, the investor is uncertain about the returns of the risky assets, while he is not uncertain about the returns of the market portfolio. This leads to a shift of the optimal portfolio towards the world market portfolio. The replicated results show that in case $\omega =$

0, already an increase in $\epsilon_a$ from 0 to 0.25 leads to an optimal portfolio, which equals the market portfolio and the Sharpe ratio is 0.1156. Given that $\epsilon_b = 0$ and $\omega = 1$, the investor prefers to invest all his wealth in the world market portfolio, even if he is certain about the asset returns ($\epsilon_a = 0$). The shift towards the world market portfolio in case that $\epsilon_b = 0$ and $\epsilon_a > 0$ is consistent with the results of GUW. However, in case that $\omega = 0$, the original results show that the optimal portfolio equals the world market portfolio only for sufficient high values of $\epsilon_a$ ($\geq$ 2.5) and otherwise the optimal portfolio consists of both, the risky assets and the world market portfolio.

If, on the other hand, $\epsilon_a = 0$ and $\epsilon_b > 0$, the investor is certain about the returns of the risky assets, but uncertain about the returns of the world market portfolio. This leads to a shift of the optimal portfolio towards a portfolio consisting only of the risky assets. Again, an increase in the ambiguity parameter $\epsilon_b$ from 0 to 0.5 is sufficient in order for the investor to prefer not to invest in the world market portfolio at all and instead to invest all his wealth in the risky assets. This is true for both cases, $\omega = 0$ and $\omega = 1$. However, in case the investor believes dogmatically in the CAPM, the Sharpe ratios are considerably higher. This result arises because the portfolio weights are less extreme if expected asset returns are determined from the CAPM instead of estimating them from the historical sample. Because of estimation errors, these extreme portfolio weights lead to a poor out-of-sample performance. The shift towards a portfolio consisting of only risky assets is consistent with the original results of GUW, the Sharpe ratios, however, differ strongly.

In the general case of $\epsilon_a > 0$ and $\epsilon_b > 0$, the investor is uncertain about the asset returns and also uncertain about the returns of the world market portfolio. The replicated results show that this leads to a situation, in which the investor doesn't want to participate in the stock market at all and instead he prefers to invest all his wealth in the risk-free asset. This result is confirmed by Boyle et al. (2012), who apply the ambiguity-averse approach as proposed by GUW and show that, if ambiguity reaches a certain level, the investor prefers to invest only in the risk-free asset. This preference, however, contradicts the construction of the portfolio optimization problem incorporating parameter and model uncertainty presented by GUW, because they don't allow for an investment in the risk-free asset. This is especially inconsistent, because a risk-free asset is introduced to determine the excess returns and for the CAPM to be valid, but the risk-free asset is not an investment option. As a result, the investor doesn't invest any wealth and the Sharpe ratio is set to 0. This result differs from the original result of GUW, who show that in the general case of $\epsilon_a > 0$ and $\epsilon_b > 0$, the investor chooses a portfolio that consists of both, the risky assets and the world market portfolio.

GUW showed in their empirical study that, most of the

**Table 3:** Replication with parameter and model uncertainty

This table reports the out-of-sample Sharpe ratio for the returns on different portfolio optimization strategies. The parameter $\epsilon_a$ presents the ambiguity about the asset returns and varies between 0 and 3 and $\epsilon_b$ presents the ambiguity about the returns of the factor portfolio, also varying between 0 and 3. The parameter $\omega$ presents the investor's beliefs in the validity of the CAPM, where $\omega = 0$ means the investor doesn't believe in the CAPM and $\omega = 1$ means the investor believes dogmatically in the CAPM. The values for the Sharpe ratios are expressed as percentage per month.

Results of Garlappi, Uppal and Wang

| Strategy | Sharpe ratio | | | | | | |
|---|---|---|---|---|---|---|---|
| Mean-Variance | -0.0719 | | | | | | |
| Minimum-Variance | 0.1490 | | | | | | |
| Bayes-Stein | -0.0528 | | | | | | |
| Naive Diversification | - | | | | | | |
| Ambiguity-averse | With $\omega = 0$ | | | $\epsilon_b$ | | | |
| | 0.00 | 0.50 | 1.00 | 1.50 | 2.00 | 2.50 | 3.00 |
| $\epsilon_a = 0.00$ | -0.0853 | 0.1127 | 0.1127 | 0.1127 | 0.1127 | 0.1127 | 0.1127 |
| $\epsilon_a = 0.25$ | -0.0774 | 0.1032 | 0.1043 | 0.1043 | 0.1043 | 0.1043 | 0.1043 |
| $\epsilon_a = 0.50$ | -0.0475 | 0.0824 | 0.0815 | 0.0824 | 0.0839 | 0.0842 | 0.0842 |
| $\epsilon_a = 0.75$ | -0.0113 | 0.0604 | 0.0568 | 0.0524 | 0.0477 | 0.0489 | 0.0481 |
| $\epsilon_a = 1.00$ | 0.0930 | 0.0655 | 0.0601 | 0.0566 | 0.0562 | 0.0556 | 0.0447 |
| $\epsilon_a = 1.50$ | 0.1219 | 0.1218 | 0.0814 | 0.0503 | 0.0391 | 0.0319 | -0.0435 |
| $\epsilon_a = 2.00$ | 0.1223 | 0.1252 | 0.0874 | 0.0643 | 0.0487 | 0.0341 | -0.0492 |
| $\epsilon_a = 2.50$ | 0.1239 | 0.1275 | 0.0909 | 0.0465 | 0.0126 | 0.0091 | -0.0870 |
| $\epsilon_a = 3.00$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |
| | With $\omega = 1$ | | | | | | |
| $\epsilon_a = 0.00$ | 0.1239 | 0.1202 | 0.1202 | 0.1202 | 0.1202 | 0.1202 | 0.1202 |
| $\epsilon_a = 0.25$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0383 | 0.0383 |
| $\epsilon_a = 0.50$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |
| $\epsilon_a = 0.75$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |
| $\epsilon_a = 1.00$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |
| $\epsilon_a = 1.50$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |
| $\epsilon_a = 2.00$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |
| $\epsilon_a = 2.50$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |
| $\epsilon_a = 3.00$ | 0.1239 | 0.1284 | 0.0943 | 0.0551 | 0.0197 | 0.0364 | -0.0647 |

Replication of results

| Strategy | Sharpe ratio | | | | | | |
|---|---|---|---|---|---|---|---|
| Mean-Variance | 0.0238 | | | | | | |
| Minimum-Variance | 0.1295 | | | | | | |
| Bayes-Stein | 0.0261 | | | | | | |
| Naive Diversification | 0.0994 | | | | | | |
| Ambguity-averse | With $\omega = 0$ | | | $\epsilon_b$ | | | |
| | 0.00 | 0.50 | 1.00 | 1.50 | 2.00 | 2.50 | 3.00 |
| $\epsilon_a = 0.00$ | -0.0616 | 0.0295 | -0.0295 | -0.0295 | -0.0295 | -0.0295 | -0.0295 |
| $\epsilon_a = 0.25$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 0.50$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 0.75$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 1.00$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 1.50$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 2.00$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 2.50$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 3.00$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

(Continued)

Table 3—continued

| With $\omega = 1$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\epsilon_a = 0.00$ | 0.1156 | 0.0950 | 0.0950 | 0.0950 | 0.0950 | 0.0950 | 0.0950 |
| $\epsilon_a = 0.25$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 0.50$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 0.75$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 1.00$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 1.50$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 2.00$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 2.50$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\epsilon_a = 3.00$ | 0.1156 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

time, allowing for ambiguity in portfolio optimization improves the out-of-sample performance, even if ambiguity is not included in the performance measure. In the following Chapter, several input parameters like the sample size, the degree of risk aversion and the timespan are varied to test the robustness of the results. In Chapter 5.3, only the model with parameter uncertainty is considered, because the model accounting additionally for model uncertainty always leads to an extreme result, where the investor either invests all his wealth in the world market portfolio, he invests all his wealth in the risky assets or he doesn't participate in the stock market at all.

## 5.3. Variation of the input parameters

### 5.3.1. Sample size

The first input parameter that is varied is the sample size, which is used to determine the optimal portfolio weights. GUW use a window length of 120 months. In this Chapter, the window length is varied between 30 months and 120 months. A maximum of 120 months is used, to make sure that the out-of-sample period remains unchanged. The case is considered, where there is parameter uncertainty only and short selling is allowed. Figure 2 presents the resulting Sharpe ratios (Panel A) and the turnover ratios (Panel B) for the different values of the window length.

Panel A shows that, for every window length, the Sharpe ratio of the minimum-variance portfolio is the highest, while the Sharpe ratio of the mean-variance portfolio is the lowest. Furthermore, it can be observed, that the ranking of the portfolio optimization strategies regarding the Sharpe ratio for every window length is the same. Also, there is no clear trend visible for either strategy as the window length increases. These results indicate that the original results of GUW are robust to changes in the sample size.

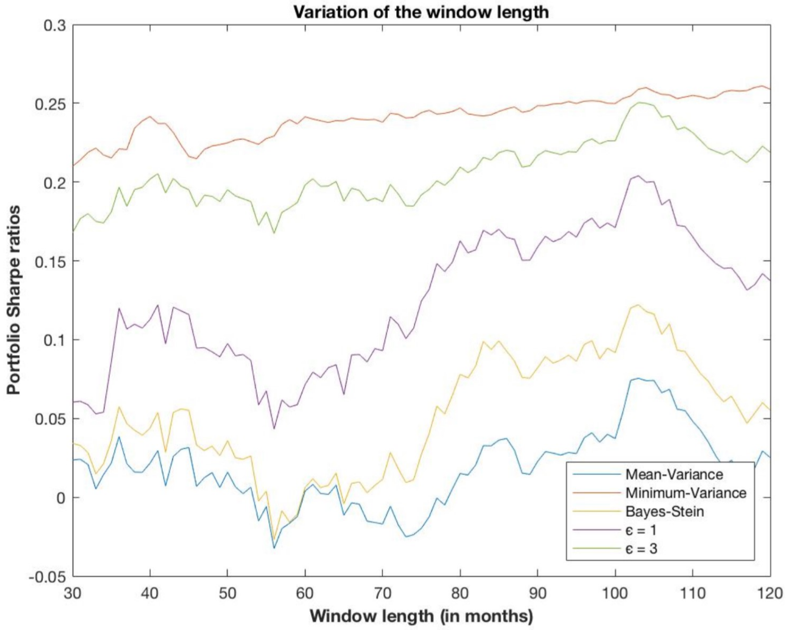
From Panel B, a clear trend becomes visible that for every strategy the portfolio turnover decreases as the window length increases. However, the effect is greatest for the mean-variance portfolio. If the sample size is large, many historical observations are used to determine the expected return and therefore, portfolio weights are less extreme and the turnover ratio decreases. The minimum-variance portfolio has the lowest turnover ratio which decreases only to a small extent as the sample size increases. This is because the minimum-variance portfolio is independent of expected returns, but also the variance-co-

variance matrix is more stable if many observations are taken into account. Consequently, the turnover ratio is not robust to the choice of the window length as it can be reduced if the window length is increased. However, the ranking of the different portfolio optimization strategies regarding the turnover ratio does not change and is consistent with the original assumption of 120 months.

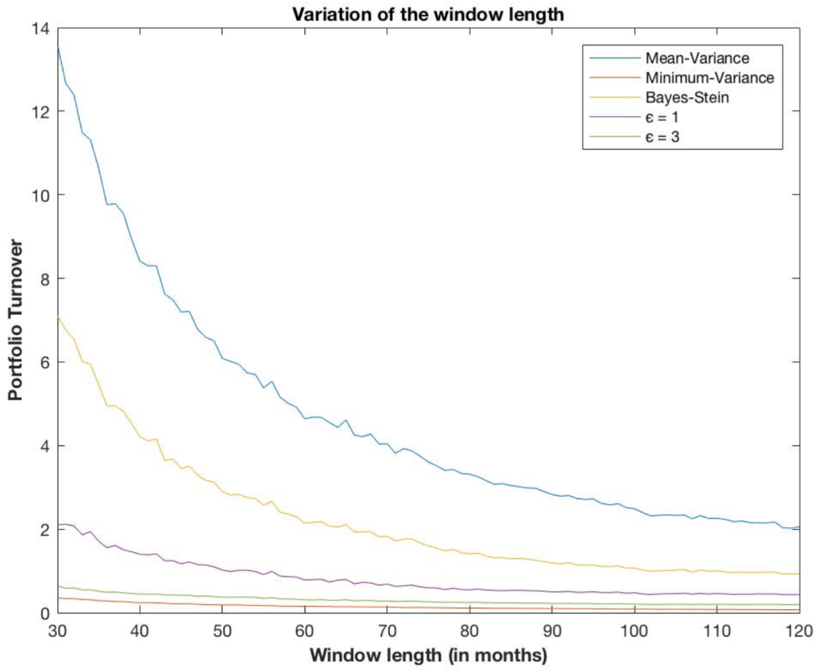
### 5.3.2. The degree of risk aversion

The investor's degree of risk aversion is presented by the variable $\gamma$. While GUW assume that $\gamma$ is fixed at 1, empirical evidence has shown that most investors have a degree of risk aversion between 2 and 4. Therefore, a value of 1 reflects a low degree of risk aversion. It has been shown that incorporating ambiguity aversion to the portfolio optimization problem leads to more conservative portfolios that are less risky than the mean-variance portfolio. This effect, however, can also be achieved by increasing the degree of risk aversion. If $\gamma$ increases, the investor prefers a portfolio further to the left on the efficient frontier, closer to the minimum-variance portfolio. Furthermore, the degree of ambiguity aversion is correlated with the degree of risk aversion (Uppal and Wang, 2003, Bossaerts et al., 2010). Scherer (2007) even argues, that it is impossible to distinguish between risk aversion and ambiguity aversion. GUW fix the degree of ambiguity aversion also at 1, however, they don't give any guidance on how the ambiguity parameter $\epsilon$ can be determined. Therefore, $\epsilon$ depends on the investor's subjective level of ambiguity about expected returns. Consequently, the assumption that investors are strongly uncertain about expected returns (high values of $\epsilon$) contradicts their low degree of risk aversion.

Since most investors have a degree of risk aversion between 2 and 4, the empirical study of GUW is repeated assuming that $\gamma = 3$. This assumption corresponds to an investor who has a moderate degree of risk aversion and therefore prefers a more conservative portfolio. In this empirical application, the case with parameter uncertainty only is considered. Table 4 presents the out-of-sample performance of the mean-variance strategy, the minimum-variance strategy,




Panel A: Sharpe ratios




Panel B: Portfolio Turnover

**Figure 2:** Variation of the window length
This figure presents the Sharpe ratios (Panel A) and the turnover ratios (Panel B) for different portfolio optimization strategies using window lengths between 30 months and 120 months. The portfolio strategies considered are the mean-variance strategy (blue line), the minimum-variance strategy (red line), the Bayes-Stein strategy (yellow line), the ambiguity-averse approach with $\epsilon = 1$ (purple line) and the ambiguity-averse approach with $\epsilon = 3$ (green line).

the Bayes-Stein strategy, the ambiguity-averse approach and the strategy of naïve diversification. Panel A shows the results if short selling is allowed, Panel B the results if short selling is not allowed.

An increase in $\gamma$ does not affect the out-of-sample performance of the minimum-variance portfolio and the uniformly diversified portfolio, since they are independent of $\gamma$. However, in Panel A, an increase in $\gamma$ leads to an increased performance of the mean-variance portfolio, as the mean increases while the standard deviation decreases strongly, leading to an improved out-of-sample Sharpe ratio. Additionally, the turnover ratio of the mean-variance portfolio is reduced sharply, because portfolio weights are less extreme as $\gamma$ increases. These effects also lead to an increase in Sharpe ratio and decrease in turnover ratio of the Bayes-Stein portfolio. As $\gamma$ gets larger, the effect of increasing the ambiguity parameter reduces, since the difference in performance between the mean-variance port-

folio and the minimum-variance portfolio gets smaller. As a result, increasing $\gamma$, when applying the classical mean-variance strategy, has a very similar effect to increasing $\epsilon$ when applying the ambiguity-averse approach. Both adjustments lead to an optimal portfolio which is shifted from the mean-variance portfolio with $\gamma = 1$ towards the minimum-variance portfolio. If $\gamma$ approaches infinity, the optimal portfolio equals the minimum-variance portfolio, as is precisely the case if $\epsilon$ approaches infinity.

If short sales are not allowed, the out-of-sample performance of the mean-variance portfolio increases only slightly as $\gamma$ increases. This is because imposing a short sale constraint has a strong effect on the performance of the mean-variance portfolio as discussed in Chapter 2.2 and Chapter 5.2.1. Since the additional constraint leads to better diversified portfolios with less extreme portfolio weights, an increase in $\gamma$, which has a similar effect, improves the out-of-sample performance only little.

Table 5 presents the Jobson-Korkie significance test of Sharpe ratios, which indicates whether the Sharpe ratios of the different portfolio optimization strategies change significantly if $\gamma$ is increased from 1 to 3. The table considers only the case when short selling is allowed, since the Sharpe ratio reacts only little to an increase in $\gamma$, if a short sale constraint is imposed.

From Table 5 it can be seen that the Sharpe ratio of the mean-variance portfolio increases significantly on the 1%-confidence level if $\gamma$ is increased from 1 to 3, while the Sharpe ratio of the Bayes-Stein portfolio increases significantly on the 5%-confidence level. The Sharpe ratios of the ambiguity-averse portfolios with $\epsilon = 1$ and $\epsilon = 3$, however, don't increase significantly. This result confirms, that increasing $\gamma$ has a similar effect on the out-of-sample performance to taking ambiguity aversion into account, when optimizing portfolios. The significance test, however, also reveals that the portfolios considering ambiguity aversion are more robust to changes in the degree of risk aversion.

Finally, Figure 3 compares the shrinkage factors of the Bayes-Stein portfolio and the ambiguity-averse portfolios with $\epsilon = 1$ and $\epsilon = 3$ in case short selling is allowed.

If $\gamma$ increases from 1 to 3, the shrinkage factor of the Bayes-Stein portfolio and the shrinkage factors of the portfolios incorporating ambiguity aversion are increasingly similar. This explains why the superiority of the ambiguity-averse portfolios over the Bayes-Stein portfolio, decreases as $\gamma$ increases. However, from Figure 3 it can be seen that the shrinkage factors of the ambiguity-averse portfolios are much more stable and that the fluctuation of the shrinkage factor decreases as $\epsilon$ increases. This result is in accordance with the results presented in Figure 1 for a degree of risk aversion of 1.

If the degree of risk aversion increases, the out-of-sample performance of the mean-variance strategy and the Bayes-Stein strategy increases significantly, while the performance of the ambiguity-averse approach increases only slightly. As a result, the superiority of the ambiguity-averse approach decreases as $\gamma$ increases. On the other hand, it was shown that the ambiguity-averse approach is much more robust to changes in the risk aversion parameter.

#### 5.3.3. Time period

GUW use in their empirical study observations from January 1970 to July 2001. In this Chapter, the timespan is extended to December 2016, while the investment universe and all other original input parameters remain unchanged. The extension of the time period indicates whether the ambiguity-averse approach outperforms other strategies, like the mean-variance approach and the Bayes-Stein approach, also if a different, more current time period is considered. As already argued in Chapter 4.2.2, historical sample paths have specific characteristics that might put an advantage to a certain model. If this advantage arises from time-specific characteristics, varying the time period is an appropriate method to test the robustness of the results over time. Firstly, a single extended time period from January 1970 to December 2016 with 564 observations is analyzed and afterwards a rolling out-of-sample period of 259 observations between January 1970 and December 2016 is considered.

Table 6 presents the out-of-sample performance of the mean-variance strategy, the minimum-variance strategy, the Bayes-Stein strategy, the ambiguity-averse approach and the strategy of naïve diversification for the time period from January 1970 to December 2016 when parameter uncertainty only is considered. Only the case when short sales are allowed is analyzed, because, analogous to the original time period, the performance of the different strategies is very similar if short sales are prohibited. Table 7 presents the Jobson-Korkie significance test of Sharpe ratios for the extended time period.

From Table 6, it can be seen that the minimum-variance portfolio outperforms all other portfolios regarding mean, standard deviation and Sharpe ratio also if the time period is extended to December 2016. However, the superiority of the minimum-variance portfolio decreases, especially regarding the mean. Table 7 shows that the Sharpe ratio of the minimum-variance portfolio isn't significantly higher than the Sharpe ratio of the mean-variance portfolio on the 5%-confidence level. The z-statistic increased from -1.9474 to -1.5978 compared to the original time period, confirming that the superiority of the minimum-variance portfo-

**Table 4:** Variation of the degree of risk aversion

This table reports the out-of-sample mean-to-standard-deviation ratio and turnover ratio for the returns on different portfolio optimization strategies. The parameter $\epsilon$ presents the ambiguity parameter and varies between 0 and infinity. The values for the means and the standard deviations are expressed as percentage per month. Turnover is expressed as shift in portfolio weights per month. The degree of risk aversion $\gamma$ is assumed to be 3.

Variation of the degree of risk aversion

| Strategy | Mean | SD | Mean/SD | Turnover |
|---|---|---|---|---|
| Panel A: Short sales allowed | | | | |
| Mean-Variance | 0.0094 | 0.0935 | 0.1006 | 0.6913 |
| Minimum-Variance | 0.0109 | 0.0421 | 0.2589 | 0.0723 |
| Bayes-Stein | 0.0095 | 0.0558 | 0.1700 | 0.3189 |
| Naive Diversification | 0.0097 | 0.0439 | 0.2214 | 0.0000 |
| Ambiguity-averse | | | | |
| $\epsilon = 0.00$ | 0.0094 | 0.0935 | 0.1006 | 0.6913 |
| $\epsilon = 0.25$ | 0.0098 | 0.0644 | 0.1516 | 0.4110 |
| $\epsilon = 0.50$ | 0.0100 | 0.0574 | 0.1738 | 0.3333 |
| $\epsilon = 0.75$ | 0.0101 | 0.0537 | 0.1885 | 0.2877 |
| $\epsilon = 1.00$ | 0.0102 | 0.0513 | 0.1990 | 0.2568 |
| $\epsilon = 1.50$ | 0.0103 | 0.0486 | 0.2126 | 0.2173 |
| $\epsilon = 2.00$ | 0.0104 | 0.0472 | 0.2209 | 0.1927 |
| $\epsilon = 2.50$ | 0.0105 | 0.0462 | 0.2265 | 0.1757 |
| $\epsilon = 3.00$ | 0.0105 | 0.0456 | 0.2305 | 0.1633 |
| $\epsilon \to \infty$ | 0.0109 | 0.0421 | 0.2589 | 0.0723 |
| Panel B: Short sales not allowed | | | | |
| Mean-Variance | 0.0103 | 0.0510 | 0.2014 | 0.1668 |
| Minimum-Variance | 0.0107 | 0.0415 | 0.2568 | 0.0481 |
| Bayes-Stein | 0.0103 | 0.0425 | 0.2427 | 0.0860 |
| Naive Diversification | 0.0097 | 0.0439 | 0.2214 | 0.0000 |
| Ambiguity-averse | | | | |
| $\epsilon = 0.00$ | 0.0103 | 0.0510 | 0.2014 | 0.1668 |
| $\epsilon = 0.25$ | 0.0103 | 0.0444 | 0.2326 | 0.1074 |
| $\epsilon = 0.50$ | 0.0104 | 0.0433 | 0.2394 | 0.0923 |
| $\epsilon = 0.75$ | 0.0104 | 0.0428 | 0.2430 | 0.0838 |
| $\epsilon = 1.00$ | 0.0104 | 0.0426 | 0.2452 | 0.0783 |
| $\epsilon = 1.50$ | 0.0105 | 0.0422 | 0.2478 | 0.0715 |
| $\epsilon = 2.00$ | 0.0105 | 0.0421 | 0.2493 | 0.0674 |
| $\epsilon = 2.50$ | 0.0105 | 0.0420 | 0.2504 | 0.0646 |
| $\epsilon = 3.00$ | 0.0105 | 0.0419 | 0.2511 | 0.0625 |
| $\epsilon \to \infty$ | 0.0107 | 0.0415 | 0.2568 | 0.0481 |

lio decreased. The ranking of the portfolio optimization strategies regarding their out-of-sample Sharpe ratio doesn't change if the extended time period is considered and, therefore, the ambiguity-averse portfolios still outperform the mean-variance portfolio and also the Bayes-Stein portfolio for a sufficient high degree of ambiguity ($\epsilon > 0.25$). However, compared to the original time period, for which the ambiguity-averse portfolios showed a significantly higher Sharpe ratio than the mean-variance portfolio on the 5%-confidence level, this isn't true anymore for the extended time period. In general, the results from Chapter 5.2.1 can be confirmed if the time period is extended, however, the effect of accounting for ambiguity aversion is stronger for the original time period.

Additionally, to extending the time period, a rolling out-of-sample period of 259 observations is considered. Beginning with the original starting date in January 1970, the starting date is shifted by one month until the end of the time period in December 2016 is reached. This approach results in 186 different out-of-sample periods with 259 observations each. The Sharpe ratios of the different portfolio optimization strategies for the rolling out-of-sample periods are presented in Figure 4.

Figure 4 shows, that the Sharpe ratio of the minimum-variance portfolio exceeds the Sharpe ratio of the mean-variance portfolio for every out-of-sample period considered.

**Table 5:** Jobson-Korkie significance test for different degrees of risk aversion

This table reports the values of the z-statistics according to the significance test of Sharpe ratios suggested by Jobson and Korkie (1981) and corrected by Memmel (2003). The significance of Sharpe ratios for the returns on different portfolio optimization strategies for different degrees of risk aversion ($\gamma = 1$ and $\gamma = 3$) is measured. The Sharpe ratios of the mean-variance strategy (MV), the Bayes-Stein strategy (BS), the ambiguity-averse approach with $\epsilon = 1$ ($\epsilon = 1.00$) and the ambiguity-averse approach with $\epsilon = 3$ ($\epsilon = 3.00$) are tested. One star * means that the p-value indicates that the Sharpe ratios are significantly different on the 5%-confidence level, while two stars ** indicate that the Sharpe ratios are significantly different on the 1%-confidence level.

Jobson-Korkie significance test

| | | | $\gamma = 3$ | | |
|---|---|---|---|---|---|
| | Strategy | MV | BS | $\epsilon = 1.00$ | $\epsilon = 3.00$ |
| | Mean-Variance | -2.7077** | | | |
| $\gamma = 1$ | Bayes-Stein | | -2.5082* | | |
| | $\epsilon = 1.00$ | | | -1.8255 | |
| | $\epsilon = 3.00$ | | | | -1.3722 |

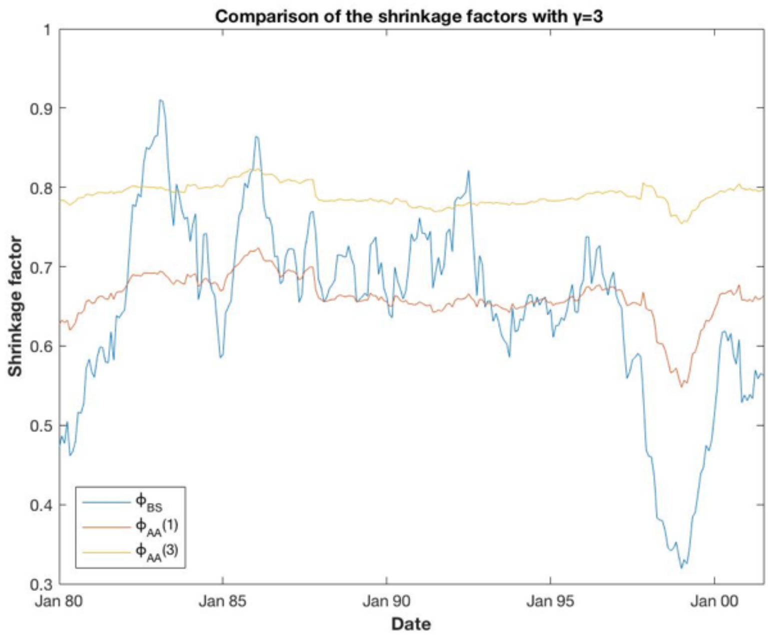



**Figure 3:** Comparison of the shrinkage factors with $\gamma = 3$

This figure compares the shrinkage factors of the Bayes-Stein portfolio and two portfolios incorporating ambiguity aversion. The blue line shows the shrinkage factor of the Bayes-Stein approach, the red line shows the shrinkage factor if the ambiguity parameter $\epsilon$ is set to 1 and the yellow line shows the shrinkage factor if the ambiguity parameter $\epsilon$ is set to 3. The degree of risk aversion $\gamma$ is assumed to be 3.

This confirms the result, that estimation errors regarding expected returns are so high, that the performance can be improved by ignoring them. In general, the results from Figure 4 are in line with the results from the previous Chapters. It is, however, important to notice that although the minimum-variance portfolio shows the highest Sharpe ratio in most cases, this is not true for every out-of-sample period. The minimum-variance approach is not able to consistently outperform the 1/N rule, while the ambiguity-averse portfolio only shows a higher Sharpe ratio than the minimum-variance portfolio for very high values of $\epsilon$ and only for few out-of-sample periods. From Figure 4 it can also be observed that the Sharpe ratios of the minimum-variance portfolio, the uniformly diversified portfolio and the ambiguity-averse portfolio with $\epsilon = 3$ for most out-of-sample periods are very similar. It is intuitive that the Sharpe ratios of the minimum-variance portfolio and the ambiguity-averse portfolio with $\epsilon = 3$ are similar, since the ambiguity-averse portfolio is strongly shrunk towards the minimum-variance portfolio if $\epsilon$ is large. The strong performance of the uniformly diversified portfolio, however, indicates that estimation errors regarding expected asset moments are so large, that an approach that completely ignores the historical sample can't be consistently outperformed by more sophisticated models.

In Chapter 5.3, different input parameters were varied to test the robustness of the original results presented by GUW. It was shown that the ambiguity-averse approach outperforms the mean-variance strategy and the Bayes-Stein strat-

**Table 6:** Variation of the time period

This table reports the out-of-sample mean-to-standard-deviation ratio and turnover ratio for the returns on different portfolio optimization strategies. The parameter $\epsilon$ presents the ambiguity parameter and varies between 0 and infinity. The values for the means and the standard deviations are expressed as percentage per month. Turnover is expressed as shift in portfolio weights per month. The degree of risk aversion $\gamma$ is assumed to be 1. The timespan considered extends from January 1970 to December 2016.

Variation of the time period

| Strategy | Mean | SD | Mean/SD | Turnover |
|---|---|---|---|---|
| Mean-Variance | 0.0074 | 0.2729 | 0.0272 | 2.6701 |
| Minimum-Variance | 0.0086 | 0.0401 | 0.2150 | 0.0878 |
| Bayes-Stein | 0.0060 | 0.1353 | 0.0444 | 1.3517 |
| Naive Diversification | 0.0074 | 0.0468 | 0.1589 | 0.0000 |
| Ambiguity-averse | | | | |
| $\epsilon = 0.00$ | 0.0074 | 0.2729 | 0.0272 | 2.6701 |
| $\epsilon = 0.25$ | 0.0063 | 0.1477 | 0.0426 | 1.4684 |
| $\epsilon = 0.50$ | 0.0067 | 0.1079 | 0.0621 | 1.0450 |
| $\epsilon = 0.75$ | 0.0072 | 0.0849 | 0.0850 | 0.7798 |
| $\epsilon = 1.00$ | 0.0076 | 0.0709 | 0.1073 | 0.6052 |
| $\epsilon = 1.50$ | 0.0080 | 0.0570 | 0.1407 | 0.4145 |
| $\epsilon = 2.00$ | 0.0082 | 0.0512 | 0.1602 | 0.3245 |
| $\epsilon = 2.50$ | 0.0083 | 0.0484 | 0.1716 | 0.2748 |
| $\epsilon = 3.00$ | 0.0084 | 0.0468 | 0.1790 | 0.2433 |
| $\epsilon \to \infty$ | 0.0086 | 0.0401 | 0.2150 | 0.0878 |

**Table 7:** Jobson-Korkie significance test - extended time period

This table reports the values of the z-statistics according to the significance test of Sharpe ratios suggested by Jobson and Korkie (1981) and corrected by Memmel (2003). The Sharpe ratios of the mean-variance strategy (MV), the minimum-variance strategy (MIN), the Bayes-Stein strategy (BS), the 1/N rule (ND), the ambiguity-averse approach with $\epsilon = 1$ ($\epsilon = 1.00$) and the ambiguity-averse approach with $\epsilon = 3$ ($\epsilon = 3.00$) are tested. One star * means that the p-value indicates that the Sharpe ratios are significantly different on the 5%-confidence level, while two stars ** indicate that the Sharpe ratios are significantly different on the 1%-confidence level. The timespan considered extends from January 1970 to December 2016.

Jobson-Korkie significance test

| Strategy | MV | MIN | BS | ND | $\epsilon = 1.00$ | $\epsilon = 3.00$ |
|---|---|---|---|---|---|---|
| Mean-Variance | 1.0000 | -1.5978 | -0.6807 | -1.0927 | -1.7562 | -1.9342 |
| Minimum-Variance | | 1.0000 | 1.5949 | -1.0656 | 1.2720 | 0.7659 |
| Bayes-Stein | | | 1.0000 | 1.0185 | -2.2208* | -2.0054* |
| Naive Diversification | | | | 1.0000 | 0.5471 | -0.2936 |
| $\epsilon = 1.00$ | | | | | 1.0000 | -1.7001 |
| $\epsilon = 3.00$ | | | | | | 1.0000 |

egy (for $\epsilon > 0.5$) regarding all settings. The superiority of the ambiguity-averse approach, however, decreases as the degree of risk aversion increases and if the time period is extended to December 2016. The ambiguity-averse approach was only able to outperform the minimum-variance approach for very few settings, when a different out-of-sample period was considered. Moreover, the ambiguity-averse portfolio only showed a higher Sharpe ratio than the uniformly diversified portfolio for very high values of $\epsilon$. In accordance with the original results, the minimum-variance portfolio showed the highest mean, lowest standard deviation and consequently also the highest Sharpe ratio for almost all settings. Since the portfolios, incorporating a sufficient high degree of ambiguity aversion, are strongly shrunk towards the minimum-variance portfolio, this explains their good performance in all settings. Furthermore, it was shown that the performance of the ambiguity-averse approach is more robust to changes in the empirical design than the mean-variance approach and the Bayes-Stein approach. The performance of the minimum-variance strategy and the 1/N rule, however, is even more robust to changes in input parameters, especially because they're both independent of the degree of risk aversion. The 1/N rule is additionally independent of the sample size.

While in this Chapter several input parameters were varied, the investment universe remained unchanged. The following Chapter deals with a portfolio optimization problem, in which the investor has the possibility to invest in the German DAX30 stocks. By changing the investment universe,

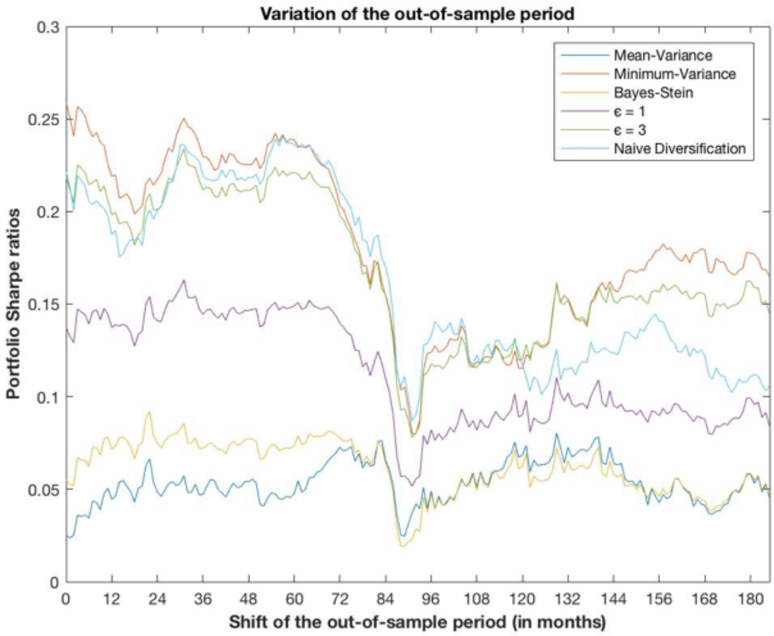



**Figure 4:** Variation of the out-of-sample period
This figure presents the Sharpe ratios for the returns on different portfolio optimization strategies for 186 different out-of-sample periods. The starting date varies between January 1970 and July 1985. The x-axis shows how many months the starting date differs from the original starting date in January 1970. The sample size to determine the portfolio weights is assumed to be 120 months. The portfolio strategies considered are the mean-variance strategy (dark blue line), the minimum-variance strategy (red line), the Bayes-Stein strategy (yellow line), the ambiguity-averse approach with $\epsilon = 1$ (purple line), the ambiguity-averse approach with $\epsilon = 3$ (green line) and the 1/N rule (light blue line

the robustness of the different portfolio optimization strategies is tested when the special characteristics of the original investment universe don't apply.

### 5.4. Application to the DAX30 stocks

#### 5.4.1. Empirical setting

In this part of the empirical analysis, the investment universe is changed and the investor builds his portfolio from the German DAX30 stocks. The returns for the time period January 1973 to December 2016 are considered. From the 30 DAX30 companies, only 16 companies were a constituent of the DAX30 over the whole time period. Consequently, the investment universe consists of these 16 German stocks (Allianz, BASF, BMW, Bayer, Beiersdorf, Commerzbank, Continental, Deutsche Bank, E-ON, Heidelberg Cement, Linde, Deutsche Lufthansa, Münchener Rück, RWE, Siemens, ThyssenKrupp). All other assumptions are taken from GUW. Portfolio weights are determined using a rolling window of 120 months and the degree of risk aversion is set to 1. Using the original assumptions of GUW, a comparison between the original results for an investment universe of eight international equity indices and the new investment universe of 16 DAX30 stocks is straightforward.

If the investment universe is shifted from eight international equity indices to the German DAX30 stocks, two main characteristics of the portfolio optimization problem change. Firstly, instead of equity indices, now single stocks are considered. The returns of single stocks fluctuate much stronger compared to the returns of equity indices, because single stocks have a higher idiosyncratic risk in addition to the systematic market risk. As a result, diversification has a stronger effect if single stocks are considered. The second main difference is the focus on a single country. This focus leads to country-specific risks that can't be diversified away. Consequently, the standard deviation of a portfolio consisting only of stocks from one country is higher compared to an internationally diversified portfolio. On the other hand, the focus on a single country might reduce ambiguity, as argued in Chapter 3.5.1. If German investors are considered, they might be more certain about expected returns of German companies compared to international companies. Therefore, if investors are sufficiently ambiguity averse, they prefer to invest in the German market only. As a result, the focus of the investor on his home country is in line with the ambiguity aversion literature (Epstein and Miao, 2003). Koziol et al. (2011) show in their empirical study, that ambiguity aversion plays an even more important role for German investors compared to international investors. All these effects are important to be considered when the portfolio optimization problem is applied to the German market.

#### 5.4.2. Empirical results

This part presents the empirical results regarding the portfolio optimization problem in which the investor has the possibility to invest in 16 DAX30 stocks. Table 8 reports the out-of-sample performance of the mean-variance strategy, the minimum-variance strategy, the Bayes-Stein strategy, the strategy introduced by GUW taking ambiguity aversion into account and the strategy of naïve diversification. The focus of the analysis lies on the model with parameter uncertainty only and again only the case when short selling is allowed is considered, because differences in performance are small if short selling is prohibited.

Table 8 shows that the standard deviation of each strategy is higher compared to the original dataset. This result is in accordance with the theoretical intuition that portfolios that contain stocks from only one country are riskier compared to an internationally diversified portfolio and that portfolios put together from single stocks are riskier than portfolios of equity indices. It can also be observed that the portfolio turnover of all strategies is higher compared to the original dataset. This is because the returns fluctuate more heavily if single stocks are considered and therefore, the portfolio needs to be reallocated more often.

From Table 8 it can be seen that, for this dataset, the mean-variance portfolio shows the highest mean, while the minimum-variance portfolio shows the lowest standard deviation. In contrast to the original dataset, for which the minimum-variance portfolio had the highest mean, the investor now can gain higher returns by taking additional risk. This is exactly the relationship Markowitz depicted with the efficient frontier. However, to gain higher returns, the investor needs to bear a much higher risk compared to the minimum-variance portfolio and as a result, the Sharpe ratio of the minimum-variance portfolio is higher than the Sharpe ratio of the mean-variance portfolio. The difference in Sharpe ratios, however, is not significant on the 5%-confidence level as can be seen from the Jobson-Korkie significance test presented in Table 9. The Bayes-Stein portfolio has a mean which is almost as high as the one of the mean-variance portfolio, but the standard deviation is lower, leading to a higher out-of-sample Sharpe ratio. The Bayes-Stein portfolio is equivalent to a portfolio incorporating ambiguity aversion with an ambiguity parameter of about 0.25. As a result, taking high values of ambiguity into account ($\epsilon > 0.25$) improves the out-of-sample performance due to the strong shift towards the minimum-variance portfolio, which shows the highest Sharpe ratio. As in the original dataset proposed by GUW, the portfolio turnover of the minimum-variance portfolio is much lower compared to the mean-variance portfolio, because the mean-variance portfolio exhibits extreme portfolio weights and requires massive reallocations.

In comparison to the original dataset, it can be seen from Table 8 that the performance of the 1/N rule compared to the ambiguity-averse approach decreased. While for the original dataset an ambiguity parameter larger than 3 was necessary to outperform the 1/N rule regarding the Sharpe ratio, the ambiguity parameter must only equal 1 in this example. This is intuitive because portfolios of individual stocks have a higher idiosyncratic volatility than portfolios of equity indices and therefore the performance of the 1/N rule decreases.

Table 9 shows that, according to the Jobson-Korkie significance test, the out-of-sample Sharpe ratio of no strategy is significantly higher on the 5%-confidence level than the Sharpe ratio of any other strategy. Although the statistical power of the Jobson-Korkie significance test is low, it becomes clear that the superiority of the minimum-variance strategy and the ambiguity-averse approach, decreased, compared to the original dataset.

The empirical application to the German DAX30 stocks was also implemented for GUW's model incorporating both, parameter and model uncertainty. In this implementation, it is assumed that assets follow a factor structure and the investor is uncertain about the return-generating model to be the true model. The return-generating model is assumed to be the CAPM and therefore the factor portfolio is the market portfolio. With regard to this portfolio optimization problem, the investor has the possibility to invest in the market portfolio in addition to the 16 DAX30 stocks. However, similar to the original dataset presented in Chapter 5.2.2, the investor prefers not to participate in the stock market at all if he is uncertain about the stock returns ($\epsilon_a > 0$) and also uncertain about the returns of the market portfolio ($\epsilon_b > 0$). If he is only uncertain about the stock returns, he invests all his wealth in the market portfolio (for $\epsilon_a > 0.25$ in case $\omega = 0$ and for $\epsilon_a \geq 0$ in case $\omega = 1$) and if he is only uncertain about the returns of the market portfolio, he invests in a portfolio consisting of only DAX30 stocks (for $\epsilon_b > 0$). Only if $\epsilon_a \leq 0.25$, $\epsilon_b = 0$ and $\omega = 0$, the investor prefers a mixed portfolio of DAX30 stocks and the market portfolio. Consequently, allowing for ambiguity about the stock returns and the returns of the market portfolio, leads to limited market participation. Since it is not allowed to invest in the risk-free asset, the Sharpe ratio in case $\epsilon_a > 0$ and $\epsilon_b > 0$ is set to 0 and the ambiguity-averse approach is outperformed by all other portfolio optimization strategies, which show a positive Sharpe ratio. The detailed results are presented in Appendix C.

The empirical application of the portfolio optimization problem to the DAX30 stocks revealed that, in general, the original results of GUW can be confirmed. Allowing for ambiguity leads to a higher out-of-sample Sharpe ratio compared to the mean-variance strategy and the Bayes-Stein strategy (for $\epsilon > 0.25$). This is, because the ambiguity-averse approach shrinks the mean-variance portfolio stronger towards the minimum-variance portfolio than the Bayes-Stein approach and the minimum-variance portfolio showed the highest Sharpe ratio. Also with regard to the DAX30 stocks, the ambiguity-averse approach wasn't able to outperform the minimum-variance portfolio, while the performance of the ambiguity-averse approach increased compared to the 1/N rule. Moreover, when looking at the model incorporating both, parameter and model uncertainty, ambiguity aversion has such a strong effect that the investor prefers not to participate in the stock market at all. To sum up, the results of the application to the DAX30 stocks are in line with the results of the original dataset, but the strong performance of the minimum-variance approach and the ambiguity-averse approach is weakened.

**Table 8:** Application to the DAX30 stocks

This table reports the out-of-sample mean-to-standard-deviation ratio and turnover ratio for the returns on different portfolio optimization strategies. The parameter $\epsilon$ presents the ambiguity parameter and varies between 0 and infinity. The values for the means and the standard deviations are expressed as percentage per month. Turnover is expressed as shift in portfolio weights per month. The degree of risk aversion $\gamma$ is assumed to be 1. The timespan considered extends from January 1973 to December 2016.

Application to the DAX30 stocks

| Strategy | Mean | SD | Mean/SD | Turnover |
|---|---|---|---|---|
| Mean-Variance | 0.0128 | 0.4538 | 0.0282 | 3.9405 |
| Minimum-Variance | 0.0087 | 0.0550 | 0.1590 | 0.1693 |
| Bayes-Stein | 0.0127 | 0.2058 | 0.0616 | 1.6240 |
| Naive Diversification | 0.0073 | 0.0635 | 0.1147 | 0.0000 |
| Ambiguity-averse | | | | |
| $\epsilon = 0.00$ | 0.0128 | 0.4538 | 0.0282 | 3.9405 |
| $\epsilon = 0.25$ | 0.0126 | 0.2098 | 0.0603 | 1.6670 |
| $\epsilon = 0.50$ | 0.0117 | 0.1388 | 0.0842 | 1.0253 |
| $\epsilon = 0.75$ | 0.0107 | 0.1047 | 0.1025 | 0.7201 |
| $\epsilon = 1.00$ | 0.0101 | 0.0880 | 0.1152 | 0.5664 |
| $\epsilon = 1.50$ | 0.0096 | 0.0740 | 0.1296 | 0.4258 |
| $\epsilon = 2.00$ | 0.0094 | 0.0684 | 0.1370 | 0.3617 |
| $\epsilon = 2.50$ | 0.0093 | 0.0654 | 0.1415 | 0.3245 |
| $\epsilon = 3.00$ | 0.0092 | 0.0635 | 0.1445 | 0.2999 |
| $\epsilon \to \infty$ | 0.0087 | 0.0550 | 0.1590 | 0.1693 |

**Table 9:** Jobson-Korkie significance test – DAX30

This table reports the values of the z-statistics according to the significance test of Sharpe ratios suggested by Jobson and Korkie (1981) and corrected by Memmel (2003). The Sharpe ratios of the mean-variance strategy (MV), the minimum-variance strategy (MIN), the Bayes-Stein strategy (BS), the 1/N rule (ND), the ambiguity-averse approach with $\epsilon = 1$ ($\epsilon = 1.00$) and the ambiguity-averse approach with $\epsilon = 3$ ($\epsilon = 3.00$) are tested. One star * means that the p-value indicates that the Sharpe ratios are significantly different on the 5%-confidence level, while two stars ** indicate that the Sharpe ratios are significantly different on the 1%-confidence level. The timespan considered extends from January 1973 to December 2016.

Jobson-Korkie significance test

| Strategy | MV | MIN | BS | ND | $\epsilon = 1.00$ | $\epsilon = 3.00$ |
|---|---|---|---|---|---|---|
| Mean-Variance | 1.0000 | -1.0932 | -1.4745 | -0.6345 | -1.6477 | -1.4114 |
| Minimum-Variance | | 1.0000 | 0.8882 | -0.6553 | 0.5636 | 0.3180 |
| Bayes-Stein | | | 1.0000 | 0.4068 | -1.3921 | -1.1660 |
| Naive Diversification | | | | 1.0000 | -0.0044 | -0.3330 |
| $\epsilon = 1.00$ | | | | | 1.0000 | -0.8398 |
| $\epsilon = 3.00$ | | | | | | 1.0000 |

## 6. Conclusion

The classical mean-variance portfolio optimization model, introduced by Harry Markowitz in 1952, assumes that expected asset moments are given and estimation errors are ignored. Since expected returns are unknown, they are inevitably estimated with error, leading to a bad out-of-sample performance of optimized portfolios. To overcome this weakness, Bayesian approaches were introduced, that are explicitly designed to reduce estimation errors. Bayesian approaches, however, assume that the investor has a unique prior about the asset moments. In this way, it is implicitly assumed that the investor is ambiguity neutral. Empirical studies (e.g. Ellsberg (1961)), however, have shown that investors are averse to ambiguity. As a consequence, new models were developed allowing for multiple priors about expected returns and thereby accounting for ambiguity aversion. This thesis is focusing on the multi-prior approach introduced by Garlappi et al. (2007) which accounts for ambiguity regarding both, expected returns (parameter uncertainty) and the return-generating model (model uncertainty). The authors show that, using their ambiguity-averse approach, the out-of-sample performance can be improved compared to the mean-variance approach and the Bayes-Stein approach.

In Chapter 5, an empirical study was performed to test whether incorporating ambiguity aversion to portfolio optimization increases the out-of-sample performance regarding the Sharpe ratio and turnover ratio for two different datasets. Since the performance measures don't account for ambiguity, it can only be tested, whether even an investor with mean-variance preferences can benefit from taking ambiguity into account. To measure the utility of the different portfolio optimization strategies for an investor who is ambiguity averse, it is necessary to measure ambiguity objectively. Furthermore, the performance of the ambiguity-averse approach depends strongly on the ambiguity parameter $\epsilon$, while Garlappi, Uppal and Wang don't give any guidance on how the ambiguity parameter can be determined. Therefore, a general statement about the performance of the ambiguity-averse approach can only be given if ambiguity is measured objectively and if it is incorporated in the performance measure.

The empirical applications have shown that the original results of Garlappi, Uppal and Wang could approximately be replicated if the model with parameter uncertainty only is considered. For the model taking both, parameter and model uncertainty, into account, however, the results differ, since, in contrast to the original results, even a small degree of ambiguity about the risky assets and the factor portfolio makes the investor prefer not to participate in the stock market at all. Limited market participation in case that investors are ambiguity averse is intuitive and was observed by various authors in the literature. But since the empirical setting of Garlappi, Uppal and Wang doesn't allow the investor to invest in the risk-free asset, a conceptual contradiction arises. Consequently, the focus of the empirical study lies on the model incorporating parameter uncertainty only.

The variation of the input parameters revealed that the results of the ambiguity-averse approach, especially for high values of $\epsilon$, are fairly robust. If the degree of risk aversion increases, the results of the ambiguity-averse approach are comparatively stable, but the superiority over the mean-variance approach and the Bayes-Stein approach decreases. This is because increasing the degree of risk aversion has a similar effect to increasing the ambiguity parameter. At the same time, it is difficult to differentiate both concepts, since they are correlated with each other. Changing the time period and the investment universe revealed that the results presented by Garlappi, Uppal and Wang are robust to time-specific and asset-specific characteristics. However, the superiority of the ambiguity-averse approach decreased in both cases.

Despite the results that show that the ambiguity-averse approach outperforms the mean-variance approach and the Bayes-Stein approach for a sufficient high level of $\epsilon$, it is important to notice that the performance of the ambiguity-averse approach depends strongly on the performance of the minimum-variance approach. Garlappi, Uppal and Wang have shown that the ambiguity-averse portfolios are a weighted average of the mean-variance portfolio and the minimum-variance portfolio. Therefore, the approach is observationally equivalent to the Bayes-Stein approach. But since the shrinkage factor of the ambiguity-averse approach for most values of $\epsilon$ is higher, the portfolios are shrunk stronger towards the minimum-variance portfolio. Consequently, above-mentioned results are achieved due to the superiority of the minimum-variance approach regarding both datasets. It is also important to consider that the 1/N rule was outperformed by the ambiguity-averse approach for high values of $\epsilon$ only. At the same time, the turnover ratio of the uniformly diversified portfolio was always the lowest, since the portfolio must only be adjusted to price changes in the stocks and not to fluctuations in the portfolio weights.

From these results, the question arises whether simple strategies like the minimum-variance strategy or the strategy of naïve diversification should be preferred to more complex approaches, like the ambiguity-averse approach. For almost all the empirical applications, the minimum-variance portfolio showed the highest out-of-sample Sharpe ratio and the lowest standard deviation, while the turnover ratio was low. The strategy of naïve diversification outperformed the mean-variance strategy in all applications and the Bayes-Stein strategy in all applications, allowing for short selling, regarding the Sharpe ratio. Furthermore, the uniformly diversified portfolio always showed the lowest turnover ratio. DeMiguel et al. (2009) compare 14 portfolio optimization models, including the classical mean-variance model, the minimum-variance model, the Bayes-Stein model, the ambiguity-averse approach introduced by Garlappi, Uppal and Wang, and the 1/N rule and find that no model is consistently better than the 1/N rule regarding Sharpe ratio and portfolio turnover. They show that the 1/N rule can only be consistently outperformed if the sample size is very large or if the number of assets is small. These assumptions, however, contradict real-life investment situations. The performance of the 1/N rule is particularly strong, if portfolios of stocks (e.g. equity indices) are considered, as done by DeMiguel, Garlappi and Wang. It was shown in Chapter 5.4.2, that the 1/N rule, however, also performs well if individual stocks are considered.

The good performance of the minimum-variance strategy and the strategy of naïve diversification indicates that estimation errors regarding expected returns are so large, that the benefits from optimal asset allocation are overshadowed (DeMiguel et al., 2009). Regarding the ambiguity-averse approach introduced by Garlappi, Uppal and Wang, the results imply that the role of ambiguity aversion in portfolio optimization is rather small compared to the importance of estimation errors. The minimum-variance strategy and the 1/N rule are both independent of expected returns and therefore also of estimation errors regarding them. This makes both strategies easy to implement and explains their good out-of-sample performance. The 1/N rule is even completely independent of realized returns, especially also of the variance-covariance matrix, and is therefore the least complex model to implement. Since realized returns are a very poor measure of future returns, the focus in future research should lie on the improvement of the estimation of asset moments (DeMiguel et al., 2009; Elton, 1999). Consequently, the development of new measures for expected returns seems to be more promising than the development of complex portfolio optimization rules that are based on realized returns.

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